Model Establishment

1. Introduction
2. Governing Equations
3. Model development—1D

1, Introduction

With the development of high speed computers, many mathematical models for suspended sediment transport in rivers have been established. They include 1-d, quasi-2d, 2d plane, 2d vertical, quasi-3d, complete 3d. Due to the obvious advantages, such as time-saving, money-saving, and scenario optimization, etc., mathematical models have been widely used to hydraulic projects to predict flow movement, sediment transport, and channel bed variations.

1, Introduction

Functions of models

Solve practical problems

- alluvial channels processes
- sedimentation in reservoirs
- down stream erosion behind hydraulic projects
- stability of water intakes
- siltation of approaching channels and harbor basins
- sediment transport in offshore zone and estuaries, etc.

Get new findings

- Through analyzing numerical results of large numbers of alternatives, new mechanisms/theories could be found, which experiments, physical models, or data analyses cannot reach.

1, Introduction

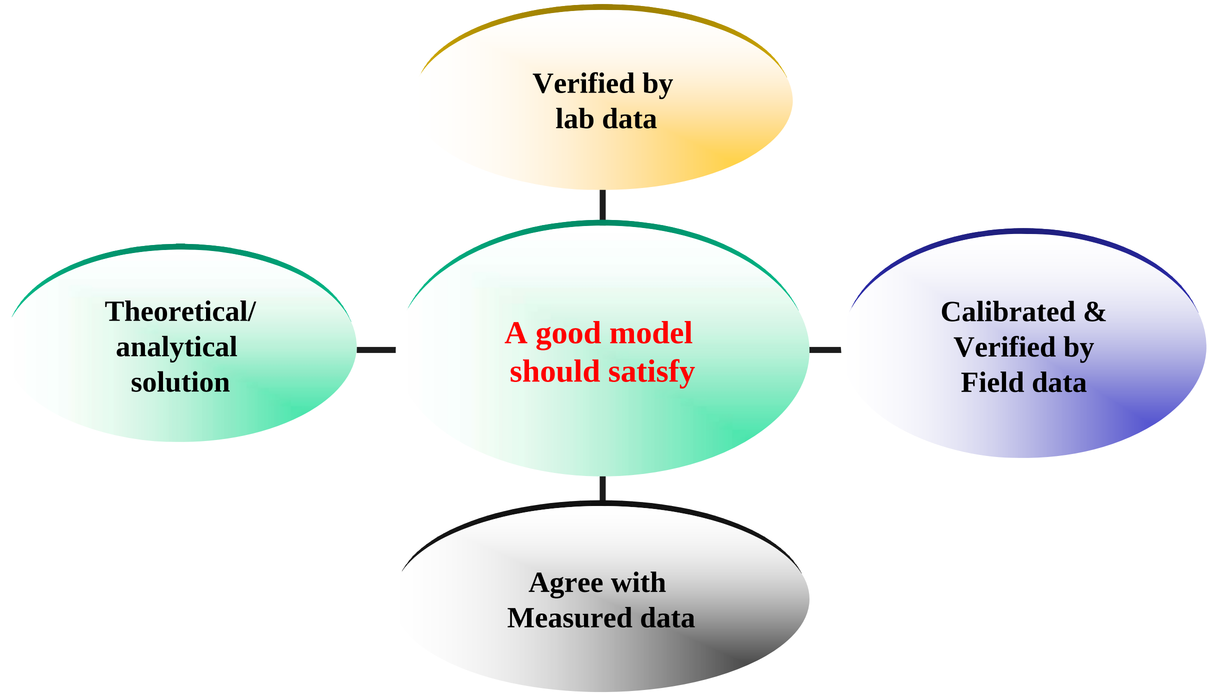
Basic requirements of mathematical models

- Satisfy physical principles
- Be verified by analytical methods,

  analytical solutions (linear)/manufactured solutions (nonlinear)
- Be validated by both experimental and field data
- Predict essential physical processes
- Be stable
- Be convergent
- Acceptable numerical results
- Agree reasonably well with physical results

1, Introduction

Basic requirements of mathematical models



Rich modeler's experience is helpful for the quality of a model

1, Introduction

Procedure of mathematical model development

Physical principles – governing equations (mainly differential eqs.)

↓

Using FDM, FEM, or other numerical methods to discretize differential eqs. to obtain discretization eqs.

↓

Write programs, test the performance of the program

↓

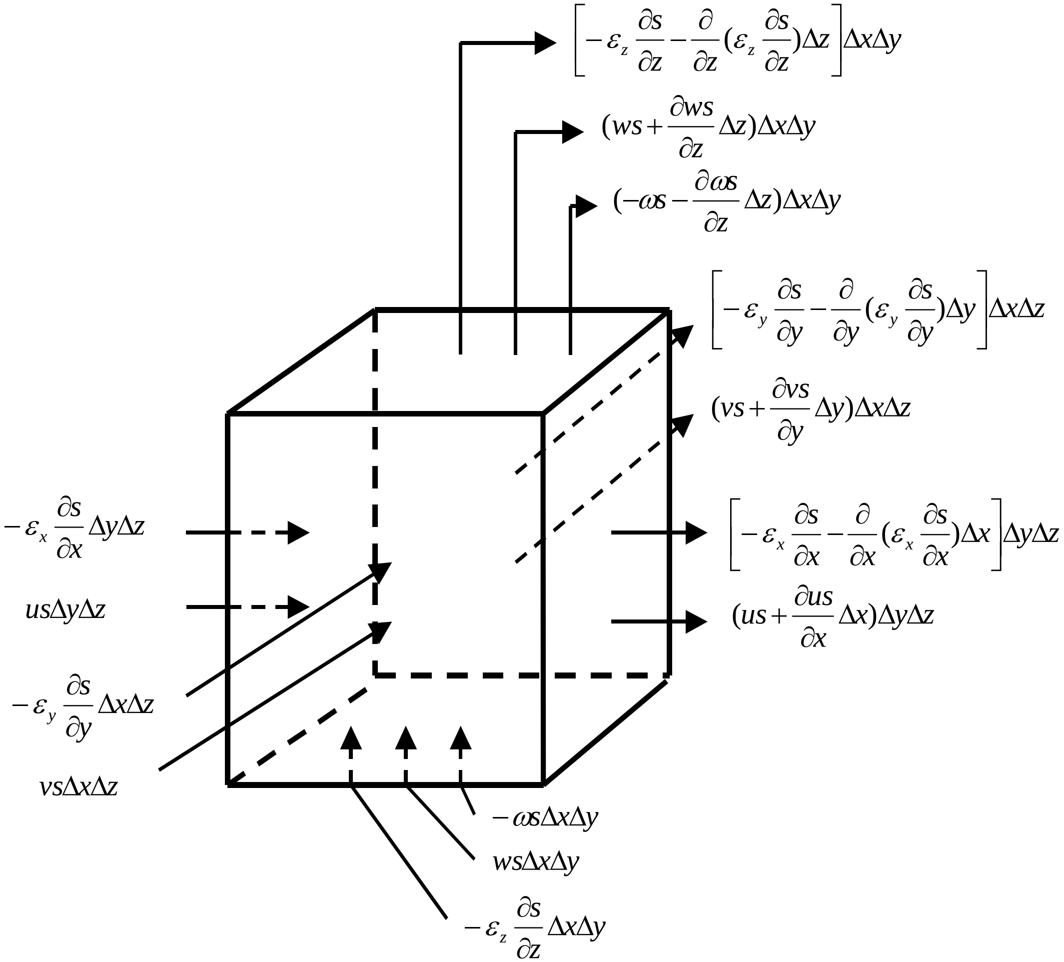
Calibration by measured data to determine related coefficients
Validation (verification) by another measured data set.

↓

Give prediction (simulation) results on requirements;
Scenario optimization; etc.

2, Governing equations

3D suspended sediment transport



2, Governing equations

3D suspended sediment transport equation

$$\frac{\partial s}{\partial t}+\frac{\partial us}{\partial x}+\frac{\partial vs}{\partial y}+\frac{\partial ws}{\partial z}=\frac{\partial \omega s}{\partial z}+\frac{\partial}{\partial x}\left(\varepsilon_x \frac{\partial s}{\partial x}\right)+\frac{\partial}{\partial y}\left(\varepsilon_y \frac{\partial s}{\partial y}\right)+\frac{\partial}{\partial z}\left(\varepsilon_z \frac{\partial s}{\partial z}\right)$$

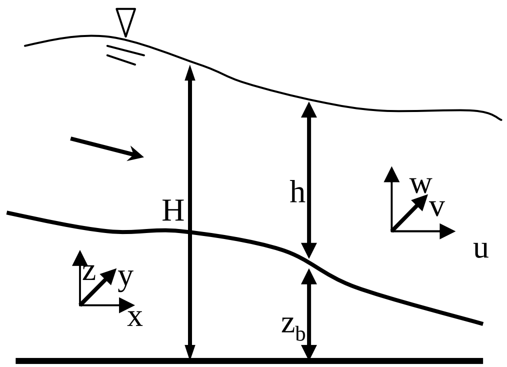
3D flow movement equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}+\frac{1}{\rho}\frac{\partial p^*}{\partial x}-\frac{1}{\rho}\left[\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}\right]=0$$

$$\frac{\partial uv}{\partial x}+\frac{\partial v^2}{\partial y}+\frac{\partial vw}{\partial z}+\frac{1}{\rho}\frac{\partial p^*}{\partial y}-\frac{1}{\rho}\left[\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}\right]=0$$

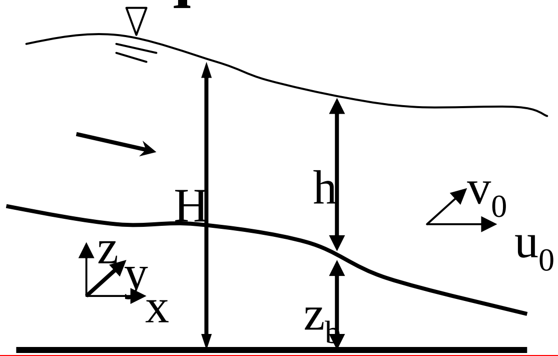
$$\frac{\partial uw}{\partial x}+\frac{\partial wv}{\partial y}+\frac{\partial w^2}{\partial z}+\frac{1}{\rho}\frac{\partial p^*}{\partial z}-\frac{1}{\rho}\left[\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right]=0$$



$$\tau_{ij}=\nu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\rho u_i' u_j'$$

2, Governing equations

2D plane flow and suspended sediment equations



$$\frac{\partial H}{\partial t}+\frac{\partial hu_0}{\partial x}+\frac{\partial hv_0}{\partial y}=0$$

$$\frac{\partial hu_0}{\partial t}+\frac{\partial hu_0u_0}{\partial x}+\frac{\partial hu_0v_0}{\partial y}-\varepsilon h\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}\right)=-gh\frac{\partial H}{\partial x}-g\frac{u_0\sqrt{u_0^2+v_0^2}}{C^2}$$

$$\frac{\partial hv_0}{\partial t}+\frac{\partial hu_0v_0}{\partial x}+\frac{\partial hv_0v_0}{\partial y}-\varepsilon h\left(\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 v_0}{\partial y^2}\right)=-gh\frac{\partial H}{\partial y}-g\frac{v_0\sqrt{u_0^2+v_0^2}}{C^2}$$

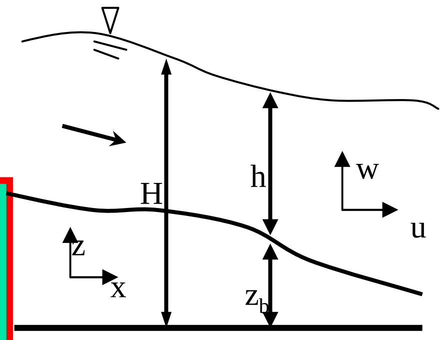
$$\frac{\partial hs_0}{\partial t}+\frac{\partial hu_0s_0}{\partial x}+\frac{\partial hv_0s_0}{\partial y}=\frac{\partial}{\partial x}\left(\varepsilon h\frac{\partial s_0}{\partial x}\right)+\frac{\partial}{\partial y}\left(\varepsilon h\frac{\partial s_0}{\partial y}\right)-\alpha\omega\left(s_0-s^*\right)$$

$$\rho'\frac{\partial z_b}{\partial t}=\alpha\omega\left(s_0-s^*\right)$$

2, Governing equations

2D vertical flow and suspended sediment equations

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0$$



$$\frac{\partial u}{\partial t}+\frac{\partial uu}{\partial x}+\frac{\partial uw}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial \sigma_x}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z}$$

$$\frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial ww}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial x}+\frac{1}{\rho}\frac{\partial \sigma_z}{\partial z}-g$$

$$\frac{\partial s}{\partial t}+\frac{\partial us}{\partial x}+\frac{\partial ws}{\partial z}=\frac{\partial \omega s}{\partial z}+\frac{\partial}{\partial x}\left(\varepsilon_x\frac{\partial s}{\partial x}\right)+\frac{\partial}{\partial z}\left(\varepsilon_z\frac{\partial s}{\partial z}\right)$$

where x and z are the horizontal and vertical coordinates, respectively; t is time; u and w are the longitudinal and vertical velocities at any location, respectively; p is the pressure; τ and σ are Reynolds turbulent shear and normal stresses; g is the gravitational accelerate; s = suspended sediment concentration; ω = the settling velocity; and ε_z = the dispersion coefficient;and ρ is the density of water.

2, Governing equations

1D flow and suspended sediment equations

Flow continuity $$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}-q_x=0$$

Flow momentum $$\frac{1}{gA}\frac{\partial Q}{\partial t}+\frac{\partial H}{\partial x}+\frac{1}{2g}\frac{\partial}{\partial x}\left(\frac{Q}{A}\right)^2+\frac{1}{g}\frac{Q}{A^2}q_x+\frac{n^2Q|Q|}{A^2R^{4/3}}=0$$

Sediment continuity $$\frac{\partial (AS)}{\partial t}+\frac{\partial (QS)}{\partial x}-s_xq_x+\alpha\omega B(S-S_*)=0$$

t,x=temporal and spatial coordinates, respectively; **A**=cross-sectional area; **Q**=volumetric flow discharge; **q_x**=lateral in/outflow, in(+),out(-); **H**=Water level; **g**=gravity acceleration; **n**=Manning roughness; **R**=hydraulic radius; **S**=sediment concentration; **s_x** = lateral inflow concentration; **S_*** =Sediment-carrying capacity; **ω**=settling velocity; **B**=cross-sectional width.

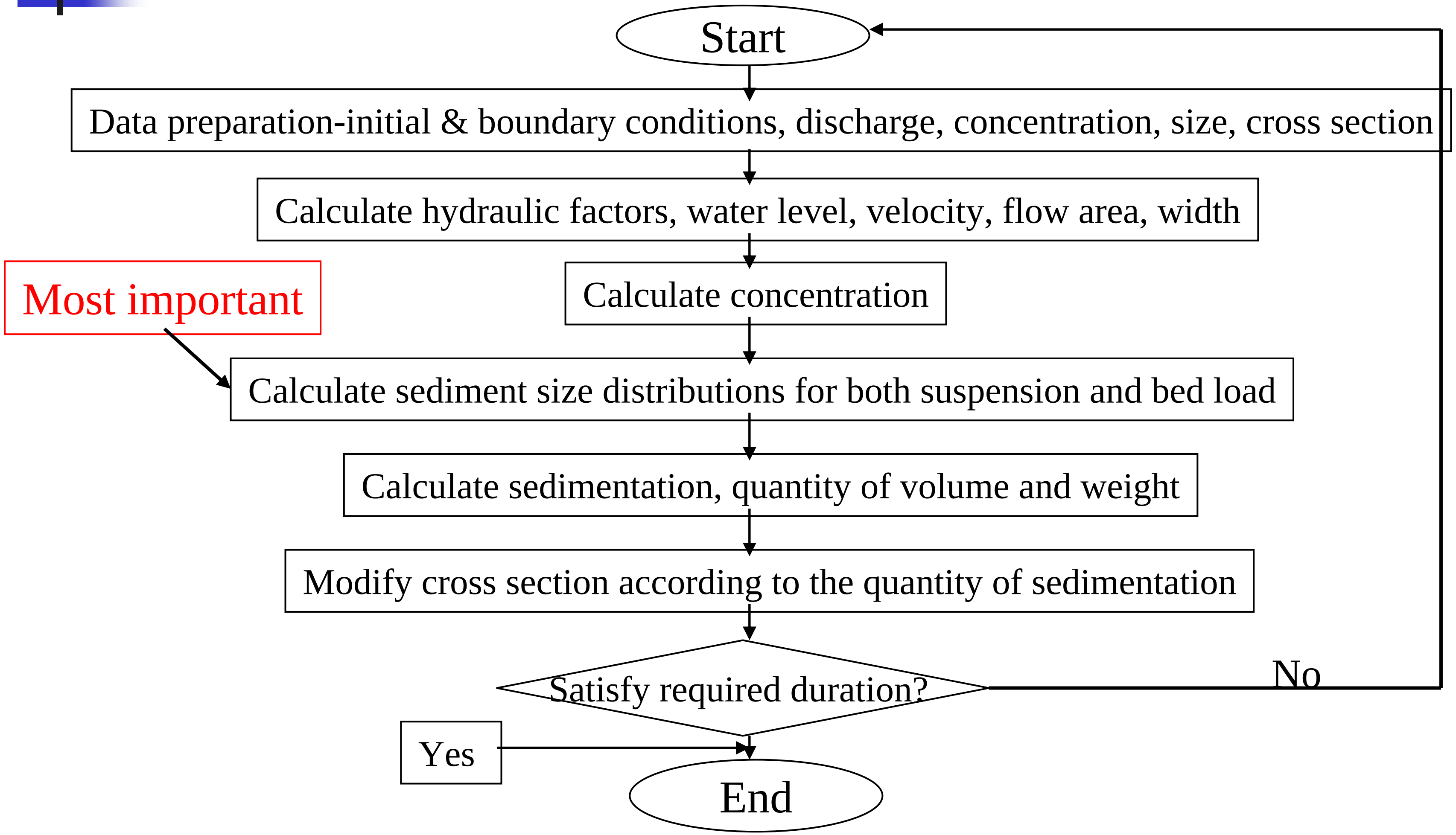
3, Model development—1D

Main Issues of UNSTM-1D

- Model Structure
- Governing Equations
- Disretization of Governing Equations
- Sediment-Carrying Capacity
- Related Coefficients
- Data Ready--Boundary and Initial Conditions
- Cross Section Information, Z~A/B
- Manning Roughness
- Treatment of non-uniform sediment
- Modification of cross sections

3, Model development—1D

3.1 Model structure



3, Model development—1D

3.2 Governing Equations

1D steady flow and suspended sediment equations

Flow, $$\frac{\partial H}{\partial x} + \frac{1}{2g}\frac{\partial U^2}{\partial x} + \frac{U^2}{C^2 R} = 0$$

Sediment, $$\frac{\partial hUS}{\partial x} = -\alpha\omega\left(S - S^*\right)$$

Bed Deformation, $$\rho'\frac{\partial Z_b}{\partial t} = \alpha\omega\left(S - S^*\right)$$

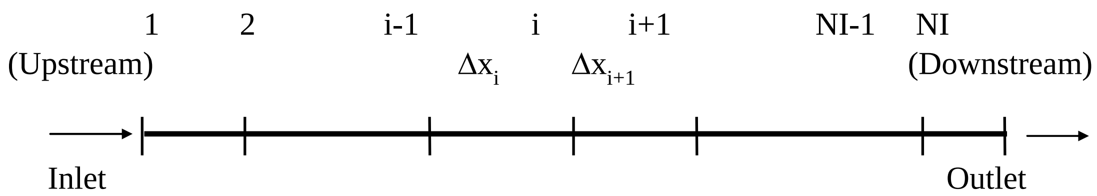
H=Water level; **U**=Velocity; **S**=Sediment concentration; **S^***=Sediment-carrying capacity; **ω**=settling velocity; **R**=Hydraulic radius; **C**=Chezy coefficient;**ρ'**=dry density; **h** = Flow depth; and **Z_b**= Channel bed elevation.

3, Model development—1D

3.3 Discretization of Governing Equations

Continuity Eq. $Q = UA$

Momentum Eq. $$\frac{\partial H}{\partial x} + \frac{1}{2g}\frac{\partial U^2}{\partial x} + \frac{U^2}{C^2 R} = 0$$



Using FDM to discretize the differential equation results in:

$$H_i = H_{i+1} + \frac{\Delta x_i n^2}{2}\left(\frac{B_i^{4/3} Q_i^2}{A_i^{10/3}} + \frac{B_{i+1}^{4/3} Q_{i+1}^2}{A_{i+1}^{10/3}}\right) + \frac{1}{2g}\left(\frac{Q_{i+1}^2}{A_{i+1}^2} - \frac{Q_i^2}{A_i^2}\right)$$

Water surface can be solved by the above FD equation. Flow area, water surface width, averaged cross-sectional velocity, etc. can then be computed.

3, Model development—1D

3.3 Discretization of Governing Equations

$$\text{Sediment transport,}\quad \frac{\partial hUS}{\partial x} = -\alpha\omega\left(S - S^*\right)$$

This equation can be solved by FDM. The algebraical solution can be easily obtained because of its simplification. Assuming $q = hU$, the equation becomes:

$$\frac{d(S-S^*)}{dx} = -\alpha\frac{\omega}{q}(S-S^*) - \frac{dS^*}{dx}$$

Its general solution can be written as:

$$S - S^* = e^{-\int\alpha\frac{\omega}{q}dx}\left(\int -\frac{dS^*}{dx}e^{\int\alpha\frac{\omega}{q}dx}dx + c\right)$$

The integration constant c can be determined by the boundary condition at $x = 0$. Therefore, the particular solution is:

$$S - S^* = \left(S_0 - S_0^*\right)e^{-\frac{\alpha\omega L}{q}} - e^{-\frac{\alpha\omega L}{q}}\int_0^L e^{\frac{\alpha\omega x}{q}}\frac{dS^*}{dx}dx$$

3, Model development—1D

3.3 Discretization of Governing Equations

If a linear variation for S^* along x direction is assumed, i.e., dS^*/dx = constant.

$$\frac{dS^*}{dx} = -\frac{S_0^* - S^*}{L}$$

The above particular solution can be written as:

$$S = S^* + \left(S_0 - S_0^*\right)e^{-\frac{\alpha\omega L}{q}} + \left(S_0^* - S^*\right)\frac{q}{\alpha\omega L}\left(1 - e^{-\frac{\alpha\omega L}{q}}\right)$$

For non-uniform sediment transport, sediment concentration can be obtained:

$$S_{i+1,j} = S^*_{i+1,j} + (S_{i,j} - S^*_{i,j})e^{-\frac{\alpha\omega_j \Delta x_i}{q_i}} + (S^*_{i,j} - S^*_{i+1,j})\frac{q_i}{\alpha\omega_j \Delta x_i}(1 - e^{-\frac{\alpha\omega_j \Delta x_i}{q_i}})$$

$$S_{i+1} = S^*_{i+1} + (S_i - S^*_i)\sum_{j=1}^{L} P_{i,j}e^{-\frac{\alpha\omega_j \Delta x_i}{q_i}} + (S^*_i - S^*_{i+1})\sum_{j=1}^{L} P_{i+1,j}\frac{q_i}{\alpha\omega_j \Delta x_i}(1 - e^{-\frac{\alpha\omega_j \Delta x_i}{q_i}})$$

i – cross section, ***j*** – sediment size group

3, Model development—1D

3.4 Sediment-carrying capacity

$$S^* = k_0\left[1+\left(\frac{\rho_s-\rho_0}{\rho_0\rho_s}\right)S\right]^m \frac{1}{\left(1-\frac{S}{\rho_s}\right)^{(k+1)m}}\left(\frac{U^3}{h\omega}\right)^m$$

Give ρ_s = 2650kg/m^3, ρ_0 = 1000kg/m^3 , S^* can be written as:

$$S^* = \frac{(1+0.00062264S)^m}{(1-0.00037736S)^{(k+1)m}} k_0\left(\frac{U^3}{h\omega}\right)^m$$

Where, ω is the representative settling velocity, it can be calculated by

$$\omega = \left(\sum_{j=1}^{L} p_j \omega_j^m\right)^{\frac{1}{m}}$$

3, Model development—1D

3.4 Sediment-carrying capacity

$$S^* = \frac{(1+0.00062264S)^m}{(1-0.00037736S)^{(k+1)m}} k_0 \left(\frac{U^3}{h\omega} \right)^m$$

From the above formula, it can be concluded: **(1)** the sediment -carrying capacity is related not only to the hydraulic factors (U,h) and sediment factors (ω), but also to the upstream concentration (S). **(2)** S wouldn't affect S^* if S is very low (say less than 20kg/m^3). However, with the increase of S, the influence of S on S^* becomes significant. Such characteristic of the formula just indicates that the Higher the upstream concn. S, the bigger the S^* even the hydraulic Factors (U,h) and sediment factors (ω) keep the same.

3, Model development—1D

3.5 Related coefficients

The model totally have four coefficients: α = adjustment coef. k_0 = sediment-carrying capacity coef., ***m*** = sediment-carrying capacity exponent, and ***k*** = adjustment exponent of settling velocity. Among them, some have relative constant values, the others can be Estimated by semi-theoretical/empirical formulas.

(1) Exponents ***m*** and ***k***

Both exponents m and k have constant values according to the research by Han (1987).

***m* = 0.92**, and ***k*=7.0** are generally acceptable.

3, Model development—1D

3.5 Related coefficients

(2) Coefficient ***k_0***

$$k_0 = \frac{B_r \rho_0 \rho_s}{\rho_s - \rho_0} \frac{1}{C^2}$$

Where, r_0 r_s are the density of water and sediment; C is Chezy coefficient, and B_r is another coef., it could be replaced by 0.01 (Bagnold, 1966) for laboratory experiments and 0.025 (Rubby 1933) for natural rivers.

Han (1980) used many field data from natural rivers and proposed a value of around 0.03 for k_0. In addition, the characteristic of k_0 is summarized as: for the same river, reservoir > downstream; the river with higher sediment concn. > the one with lower sediment concn.

For example, for Yangtze River, k_0=0.03 for the Three Gorges reservoir and k=0.017 for the downstream; for the Yellow River, K=0.04 for Sanmenxia reservoir and k=0.03 for the lower YR.

3, Model development—1D

3.5 Related coefficients

(2) Adjustment coefficient α

Theoretically, coef. α is the ratio of sediment concn. Near the bed to the depth-averaged concn. Under equilibrium condition. Guo and Jin (1999) gave an expression to estimate coef. α:

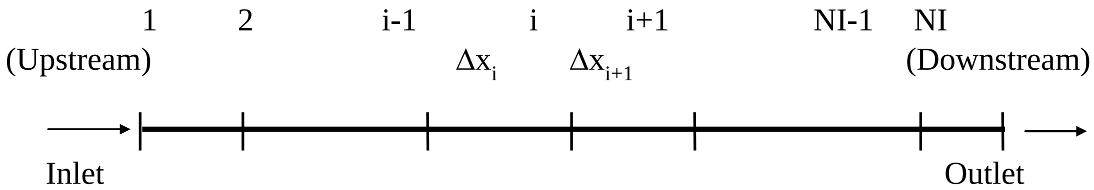
$$\alpha = \frac{S_b}{S} = \frac{\left(\frac{1}{\eta_b} - 1\right)^z}{\left[(1+\frac{\sqrt{g}}{\kappa C})\int_{\eta_b}^{1}(\frac{1}{\eta}-1)d\eta + \frac{\sqrt{g}}{\kappa C}\int_{\eta_b}^{1}(\frac{1}{\eta}-1)^z \ln\eta d\eta\right]}$$

Where, the exponent $z=\omega/\beta\kappa u_*$, and η_b=the reference relative depth. β=1.0 for $\omega/u_*<0.1$, $\beta=1+2(\omega/u_*)^2$ for $0.1<\omega/u_*<0.707$, and β=2.0 for $\omega/u_*>0.707$.

3, Model development—1D

3.6 Data ready

(1) Boundary conditions



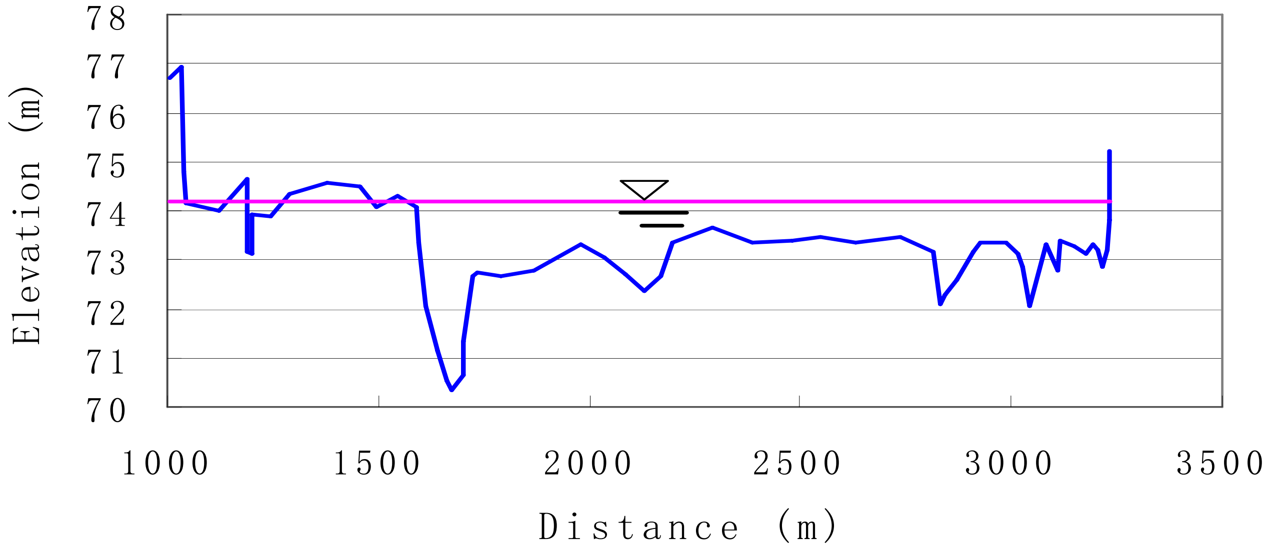
At the inlet, flow discharge Q, sediment concn. S, and sediment Size distribution P_j should be given. Meanwhile, at the outlet, The water surface level H should be given.

(2) Initial conditions

The initial size distribution of bed load and cross section information should be given.

3, Model development—1D

3.7 Cross section information



The cross-sectional area (A) and width (B) at a certain water level (z) can be expressed as follows:

$$A = f_A(z)$$

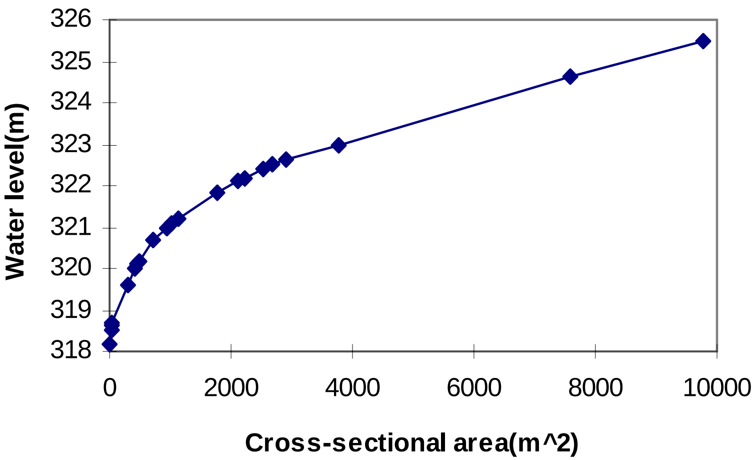
$$B = f_B(z)$$

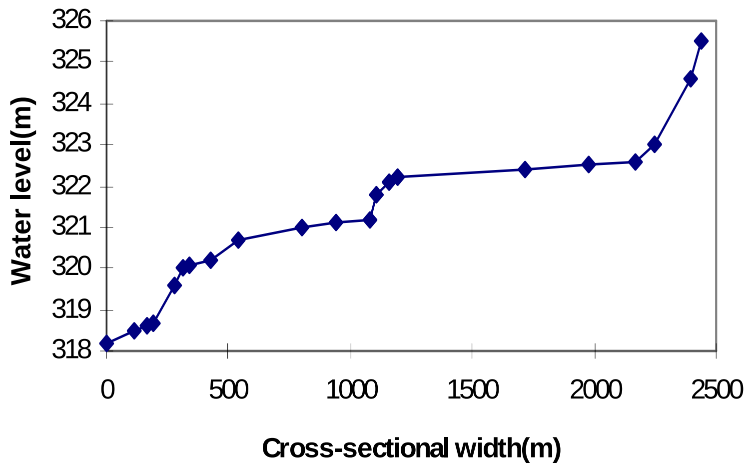
The functions of f_A and f_B could be obtained from measured data.

3, Model development—1D

3.7 Cross section information

Here is an example to show the relationships of A~H and B~H.



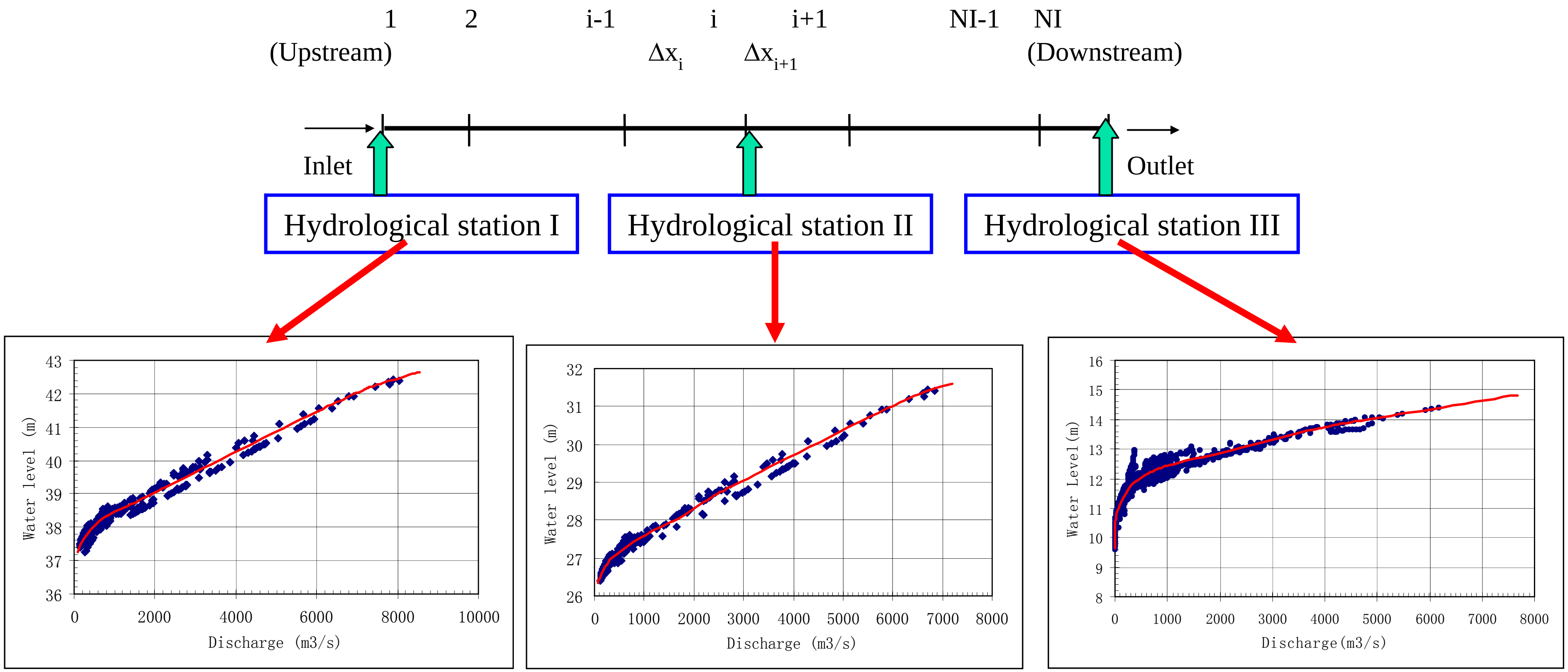


3, Model development—1D

3.8 Manning roughness

(1) Get Q~Z relation

Plot flow discharge and water level curves (Q~Z) for all stations along the computational river reach.



3, Model development—1D

3.8 Manning roughness

(2) Get Z for a certain Q

Reading water level (Z) for a certain discharge (Q) from the above Q~Z cureves results in the following table:

Q	Station I	Station II	Station III
500	38.2	27.3	12.1
1000	38.5	27.6	12.45
2000	39.0	28.3	12.8
5000	40.8	30.3	14.1
…..	…..	…..	…..

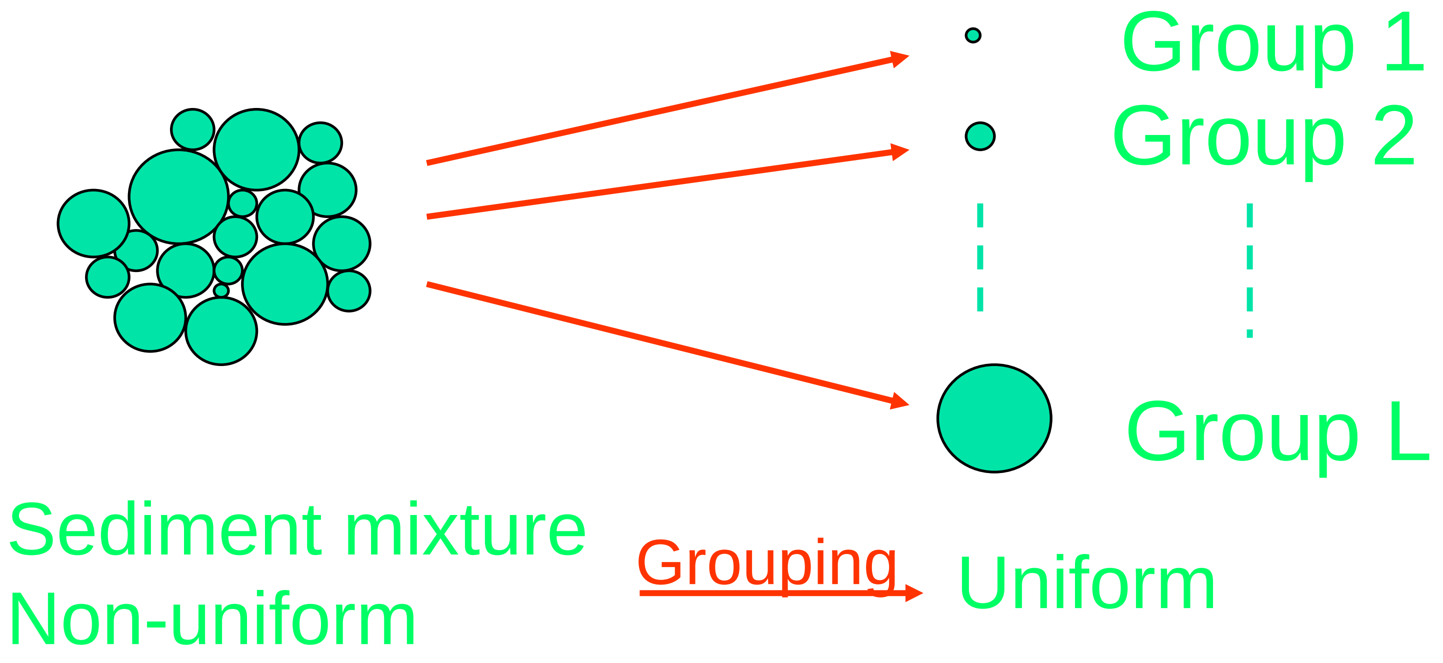
(3) Get Manning roughness *n* from the following eq. by try-error

$$H_i = H_{i+1} + \frac{\Delta x_i n^2}{2}\left(\frac{B_i^{4/3} Q_i^2}{A_i^{10/3}} + \frac{B_{i+1}^{4/3} Q_{i+1}^2}{A_{i+1}^{10/3}}\right) + \frac{1}{2g}\left(\frac{Q_{i+1}^2}{A_{i+1}^2} - \frac{Q_i^2}{A_i^2}\right)$$

3, Model development—1D

3.9 Treatment of non-uniform sediment

(1) Grouping, non-uniform ➔uniform



Non-uniform sediments are divided into several groups, in which sediment size is considered to be uniform. Therefore, theories for uniform sediment can be used in investigating non-uniform sediment

3, Model development—1D

3.9 Treatment of non-uniform sediment

(2) Calculation of suspended sediment size distribution

Deposition: $$P_{i+1,j} = P_{i,j}(1-\lambda_i)^{\left(\frac{\omega_j}{\omega_{c,i}}\right)^{\theta-1}}$$ θ = 0.75 for rivers, and 0.5 for reservoirs

Erosion $$P_{i+1,j} = \frac{1}{1-\lambda_i}\left(P_{i,j} - \frac{\lambda_i}{\lambda_i^*} R_{i,j} \lambda_i^{*\left(\frac{\omega_j}{\omega_{c,i}}\right)}\right)$$ Here : $$\lambda_i = \frac{S_i Q_i - S_{i+1} Q_{i+1}}{S_i Q_i}$$

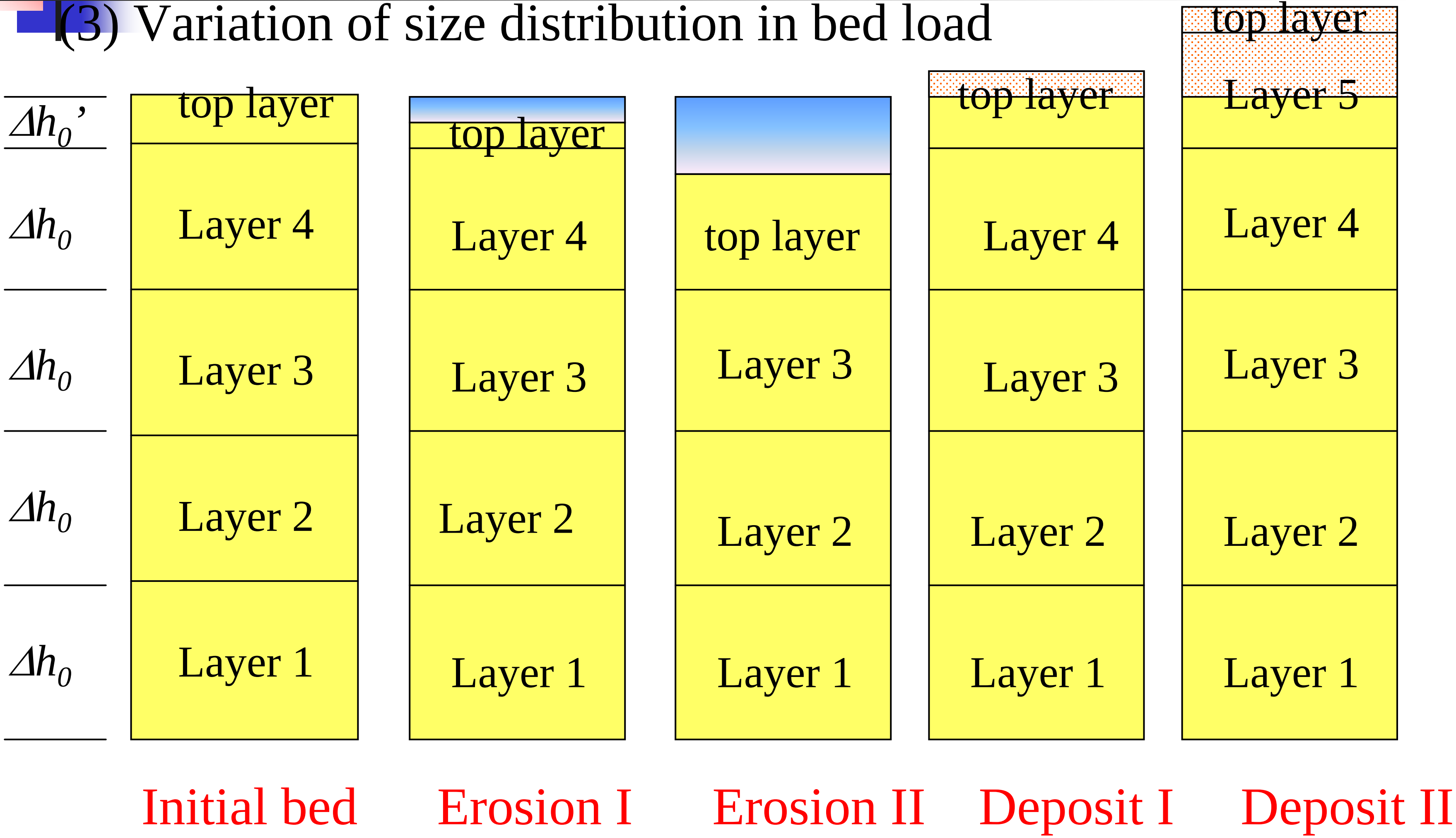
and $$\lambda_i^* = \frac{\Delta h_i'}{\Delta h_0 + \Delta h_i'}$$ Here: Δh = erosion thickness, and Δh’ = disturbing thickness.

In addition, ω_c is determined by: $$\sum_{j=1}^{L} P_{i+1,j} = 1$$

3, Model development—1D

3.9 Treatment of non-uniform sediment

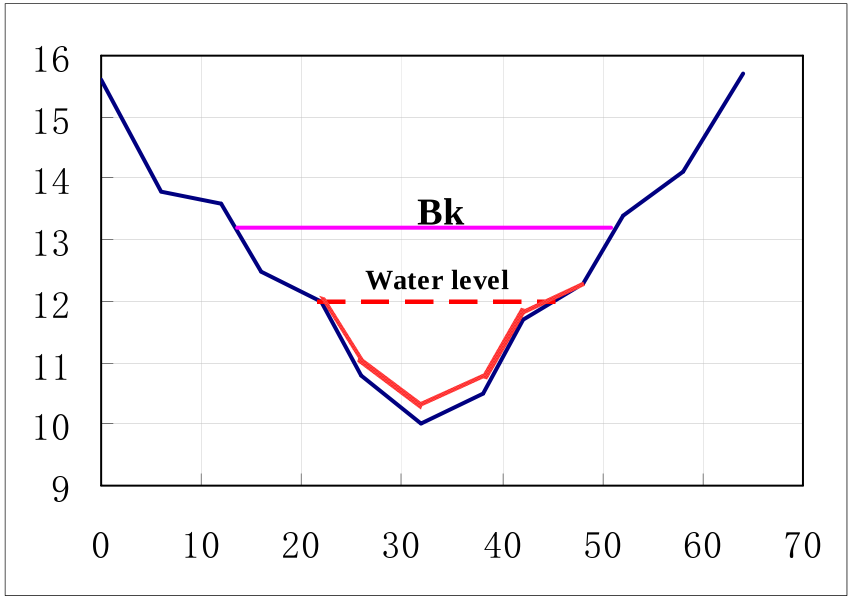
(3) Variation of size distribution in bed load



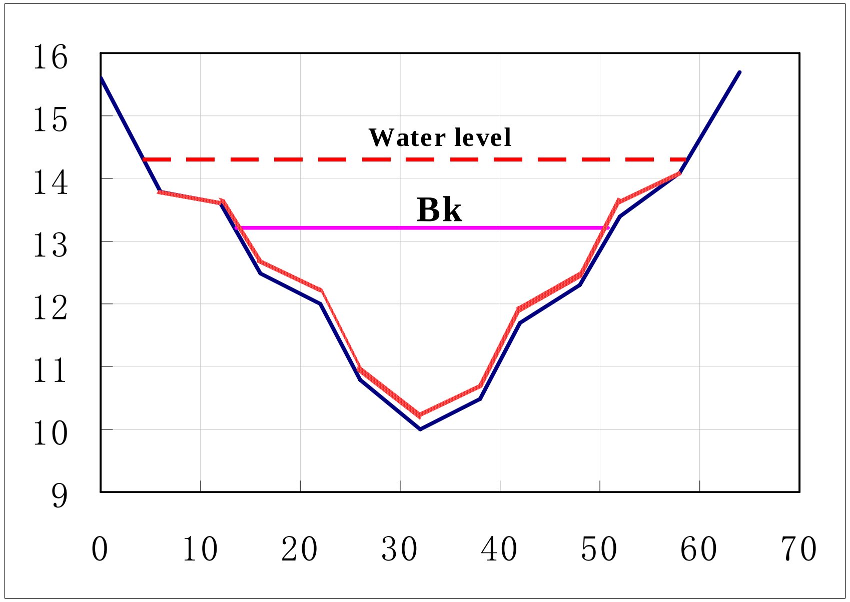
3, Model development—1D

3.10 Modification of cross sections

For the case of deposition, the deposited materials are uniformly allocated to the wetted perimeter.



Water level lower than that of Bk

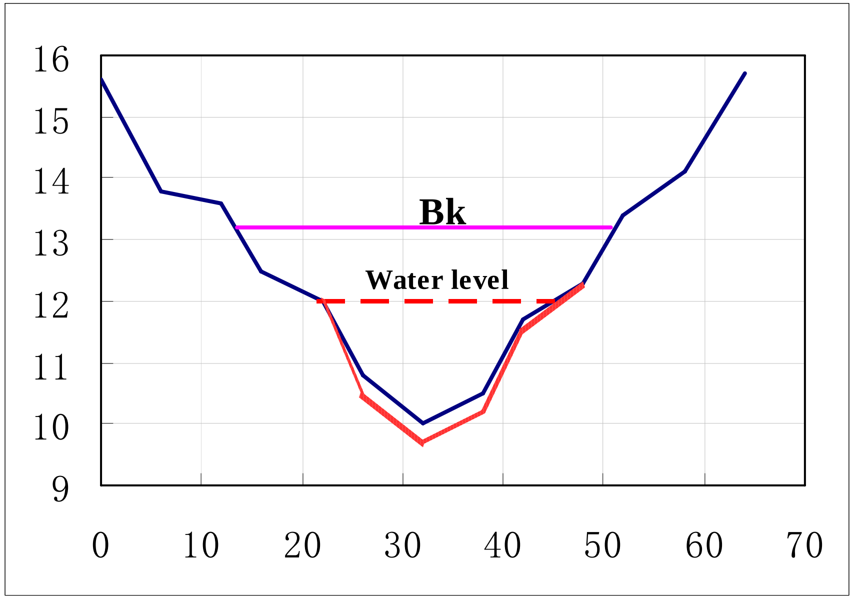


Water level higher than that of Bk

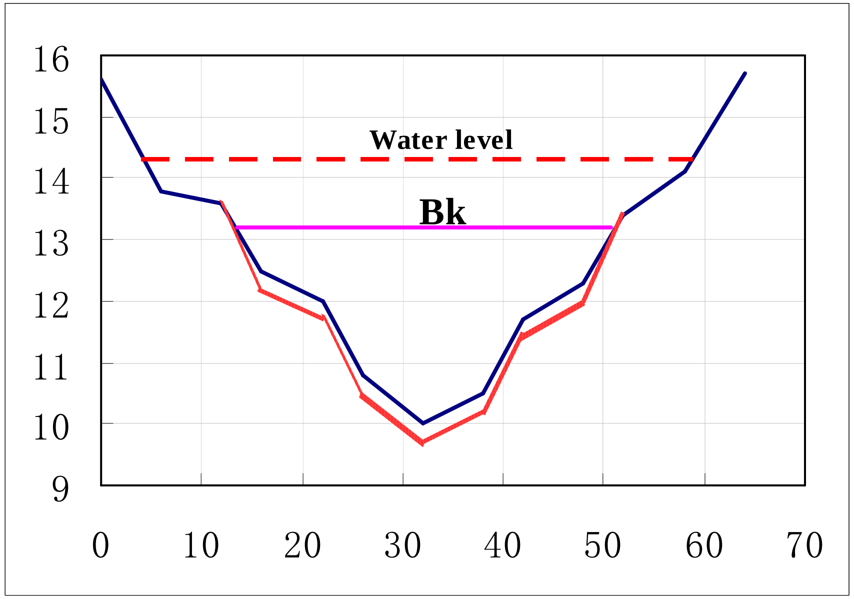
3, Model development—1D

3.10 Modification of cross sections

For erosion, when the width of flow surface is smaller than B_k, the channel bed is uniformly lowered based on the erosion amount. However, if the surface width is larger than B_k, only the channel bed below B_k is modified.



Water level lower than that of B_k



Water level higher than that of B_k

B_k is the bed-forming/bankfull width, a critical width to separate main channel & floodplain

Thanks