Sedimentation Dept., IWHR

FINITE DIFFERENCE METHOD

1. **Introduction**
2. **Discretization**
3. **Generalized formulation**
4. **Temporal Finite Difference Approximation**

FINITE DIFFERENCE METHOD

1. Introduction

- Reduction of a differential equation → algebraic equation.

Differential equation ⇐ Most governing equations are differential eqs. Most of them have no analysis solution.

↓

discretization ⇐ identify finite number of discrete points within the domain,these points are called NODES

↓

approximation ⇐ the derivatives in the governing equation are replaced by discrete differences approximation

↓

solution

- Is the approximation solution a good solution?

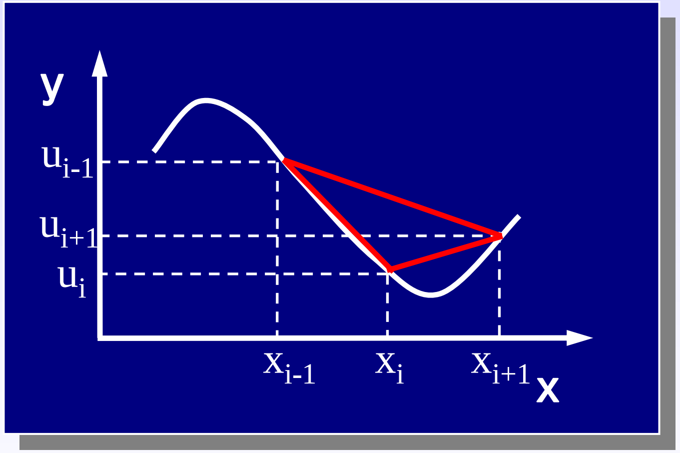
$$|u - U| < \varepsilon$$

FINITE DIFFERENCE METHOD

2. Discretization

- The definition for the derivation of a continuous function u(x) is

$$\frac{du}{dx} = \lim_{h \to 0} \frac{u(x+h) - u(x)}{(x+h) - x} = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}$$



without the limit process we can get the finite difference approximation

$$\frac{du}{dx} = \frac{u(x+h) - u(x)}{h}$$

choose the location of x and x+h to coincide with the node points

$$\left.\frac{du}{dx}\right|_{x_i} \approx \frac{u(x_i + h) - u(x_i)}{(x_i + h) - x_i} = \frac{u_{i+1} - u_i}{x_{i+1} - x_i}$$

- A derivative does not have a unique finite difference approximation. du/dx can be written as:

$$\left.\frac{du}{dx}\right|_{x_i} \approx \frac{u_{i+1} - u_i}{x_{i+1} - x_i} \quad \text{or} \quad \left.\frac{du}{dx}\right|_{x_i} \approx \frac{u_i - u_{i-1}}{x_i - x_{i-1}} \quad \text{or} \quad \left.\frac{du}{dx}\right|_{x_i} \approx \frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i-1}}$$

FINITE DIFFERENCE METHOD

2. Discretization:formulas by Taylor series expansion(i)

series expansion

$$u(x+\Delta x)=u(x)+\Delta x\frac{d}{dx}u(x)+\frac{\Delta x^2}{2!}\frac{d^2}{dx^2}u(x)+\frac{\Delta x^3}{3!}\frac{d^3}{dx^3}u(x)+\ldots\ldots\frac{\Delta x^{n-1}}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}u(x)+R_n$$

R is a remainder term defined as $R_n=\frac{\Delta x^n}{n!}\frac{d^n}{dx^n}u(x)$

An approximating finite difference expression for node point i+1 is

$$u_{i+1}=u_i+\Delta x\frac{d}{dx}u_i+\frac{\Delta x^2}{2!}\frac{d^2}{dx^2}u_i+\frac{\Delta x^3}{3!}\frac{d^3}{dx^3}u_i+R_4$$

re-arrange the equation $u_{i+1}=u_i+\Delta x\frac{du_i}{dx}+R_2$ => $\frac{du_i}{dx}=\frac{u_{i+1}-u_i}{\Delta x}+R_2$

the finite difference expression for node point i-1 is

$$u_{i-1}=u_i-\Delta x\frac{du_i}{dx}+\frac{\Delta x^2}{2!}\frac{d^2u_i}{dx^2}-\frac{\Delta x^3}{3!}\frac{d^3u_i}{dx^3}+R_4$$ => $$\frac{du_i}{dx}=\frac{u_i-u_{i-1}}{\Delta x}+R_2$$

FINITE DIFFERENCE METHOD

2. Discretization:formulas by Taylor series expansion(ii)

series expansion

$$u_{i+1} = u_i + \Delta x \frac{du_i}{dx} + \frac{\Delta x^2}{2!}\frac{d^2 u_i}{dx^2} + \frac{\Delta x^3}{3!}\frac{d^3 u_i}{dx^3} + R_4$$

$$u_{i-1} = u_i - \Delta x \frac{du_i}{dx} + \frac{\Delta x^2}{2!}\frac{d^2 u_i}{dx^2} - \frac{\Delta x^3}{3!}\frac{d^3 u_i}{dx^3} + R_4$$

the approximation to d^2u/dx^2 is obtained by elimination of du/dx between the above equations

$$\frac{d^2 u}{dx^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + R_4$$

- the finite difference approximation truncates the series,leading to the **truncation error**,**TE**,defined as the difference between the true derivative and the finite difference approximation to it.
- **TE** is due to higher-order terms in Taylor series neglected in finite difference approximations.

FINITE DIFFERENCE METHOD

2. Discretization:formulas by Taylor series expansion(iii)

- **TE** for the finite difference expressions du/dx are

$$\left.\frac{du}{dx}\right|_{x_i}-\frac{u_{i+1}-u_i}{\Delta x}=-\frac{\Delta x}{2!}\frac{d^2u_i}{dx^2}-\frac{\Delta x^2}{3!}\frac{d^3u_i}{dx^3}-R_3=O(\Delta x)$$

$$\left.\frac{du}{dx}\right|_{x_i}-\frac{u_i-u_{i-1}}{\Delta x}=-\frac{\Delta x}{2!}\frac{d^2u_i}{dx^2}+\frac{\Delta x^2}{3!}\frac{d^3u_i}{dx^3}-R_3=O(\Delta x)$$

$$\left.\frac{du}{dx}\right|_{x_i}-\frac{u_{i+1}-u_{i-1}}{2\Delta x}=-\frac{\Delta x^2}{2\bullet 3!}\frac{d^3u_i}{dx^3}-\frac{\Delta x^4}{2\bullet 5!}\frac{d^5u_i}{dx^5}-R_7=O\left(\Delta x^2\right)$$

- The order of the approximation $O(\Delta x^2)$: the lowest order tern in TE
- **TE** for d^2u/dx^2 is

$$\left.\frac{d^2u}{dx^2}\right|_{x_i}-\frac{u_{i+1}-2u_i+u_{i-1}}{\Delta x^2}=-\frac{\Delta x^2}{4!}\frac{d^4u_i}{dx^4}-\frac{\Delta x^4}{6!}\frac{d^6u_i}{dx^6}-R_8=O\left(\Delta x^2\right)$$

- the **TE** must vanish as $\Delta x \to 0$， when this is the case, the finite difference approximation is said to be consistent.

consistency requirement: $\lim_{\Delta x\to 0} TE = 0$

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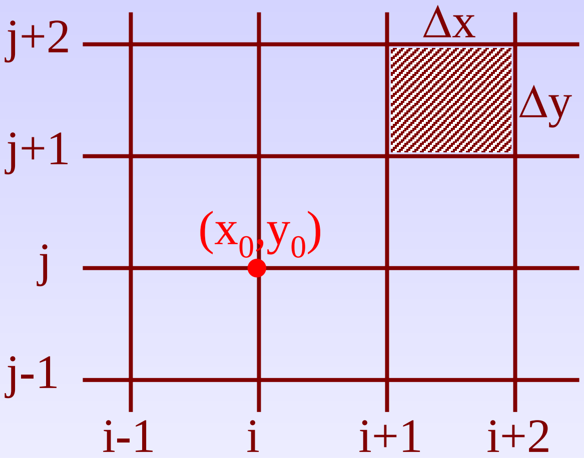
3. Generalized formulation(i)

- Replace the continuous domain by finite difference grid



$u(x,y) \rightarrow u(i\Delta x, j\Delta y)$; $0 \le x \le n\Delta x$, $0 \le y \le n\Delta y$

let $u_{i,j}$ to represent $u(i\Delta x, j\Delta y)$ or $u(x_0,y_0)$

then $u_{i+1,j} = u(x_0 + \Delta x, y_0)$

$$u_{i+1,j+1} = u(x_0 + \Delta x, y_0 + \Delta y)$$

for time dependent problems

$$u^{k+1}_{i+1,j+1} = u(x_0 + \Delta x, y_0 + \Delta y, t_0 + \Delta t)$$

- different schemes are possible,best scheme is to optimize for accuracy,economy and programming simplicity.
- from Taylor series expansion to get

Forward difference:

$$\left.\frac{\partial u}{\partial x}\right|_{x_i,y_j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x) \qquad \left.\frac{\partial u}{\partial y}\right|_{x_i,y_j} = \frac{u_{i,j+1} - u_{i,j}}{\Delta y} + O(\Delta y)$$

Backward difference:

$$\left.\frac{\partial u}{\partial x}\right|_{x_i,y_j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} + O(\Delta x) \qquad \left.\frac{\partial u}{\partial y}\right|_{x_{i+1},y_j} = \frac{u_{i+1,j} - u_{i+1,j-1}}{\Delta y} + O(\Delta y)$$

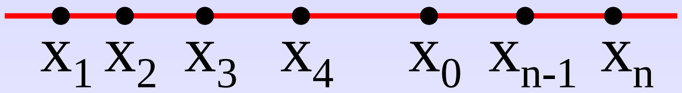
Central difference:

$$\left.\frac{\partial u}{\partial x}\right|_{x_i,y_j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O(\Delta x^2)$$

FINITE DIFFERENCE METHOD

3. Generalized formulation(ii)

- for d^2u/dx^2 $$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + O(\Delta x^2)$$
- for general one-dimensional grid, d^pu/dx^p can be obtained from



$$\left.\frac{d^p u}{dx^p}\right|_{x_i} = \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3 + \ldots\ldots = \sum_{m=1}^{n} \gamma_m u_m$$

$$\begin{Bmatrix} 1 & 1 & \ldots & 1 \\ x_1 - x_0 & x_2 - x_0 & \ldots & x_n - x_0 \\ (x_1 - x_0)^2 & (x_2 - x_0)^2 & \ldots & (x_n - x_0)^2 \\ \vdots & \vdots & \ldots & . \\ \vdots & \vdots & \ldots & . \\ (x_1 - x_0)^p & (x_2 - x_0)^p & \ldots & (x_n - x_0)^p \end{Bmatrix} \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ . \\ . \\ . \\ \gamma_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ p! \end{Bmatrix}$$

- The procedure provides consistent finite difference approximation for any order derivative on an arbitrary array of *n* node points, provided only that $n \geq p+1$.
- In general,approximation up to $O(\Delta x_{max})^{n-p}$ can be achieved.
- When n=p+1, the minimum number of points is used, and the resulting approximation will in general be first order,although second-order approximations may be obtained when Δx_1 is constant and p is even.

FINITE DIFFERENCE METHOD

Example 1: (i)

- Derive a FD approximation to the equation that describes steady state diffusion of a dissolved substance into a quiescent fluid body in which a first-order reaction occurs:

$$D\frac{d^2C}{dx^2} - KC = 0, \qquad 0 < x < 1cm$$
$$C(0) = 0, \qquad C(1) = C_1$$

C: concentration, (C_1=1 g/cm^3)

D: diffusion coefficient (=0.01 cm^2/s)

K: reaction rate (=0.1 l/s)

1. derive an approximation for d^2C/dx^2 using x_{i-1}, x_i and x_{i+1}

P=2, n=3 $$\begin{bmatrix} 1 & 1 & 1 \\ -2\Delta x & -\Delta x & 0 \\ 4\Delta x^2 & \Delta x^2 & 0 \end{bmatrix} \begin{Bmatrix} \gamma_{i-1} \\ \gamma_i \\ \gamma_{i+1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 2 \end{Bmatrix} \Rightarrow \gamma_{i-1} = \frac{1}{\Delta x^2}, \quad \gamma_i = \frac{-2}{\Delta x^2}, \quad \gamma_{i+1} = \frac{1}{\Delta x^2}$$

FD expression:

$$D\frac{C_{i+1} - 2C_i + C_{i-1}}{\Delta x^2} - KC_i = 0$$

$$C_{i+1} + \left(-\frac{K\Delta x^2}{D} - 2\right)C_i + C_{i-1} = 0$$

FINITE DIFFERENCE METHOD

Example 1: (ii)

$$C_{i+1} + \left(-\frac{K\Delta x^2}{D} - 2 \right) C_i + C_{i-1} = 0$$

- The equation is the algebraic equation used to solve for the nodal approximations

at the boundary $C_{x=0} = 0 \quad and \quad C_{x=1cm} = 1$

the equation in matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & \dots \\ 1 & -\frac{K\Delta x^2}{D} - 2 & 1 & 0 & \dots & \dots \\ 0 & 1 & -\frac{K\Delta x^2}{D} - 2 & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ . \\ C_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ . \\ 1 \end{Bmatrix}$$

re-write as

$[M]\{C\} = \{F\}$ or $M_{i,j} C_i = F_i$

FINITE DIFFERENCE METHOD

Example 1: (iii)

- Impose 3,5,10 and 20 points discretization

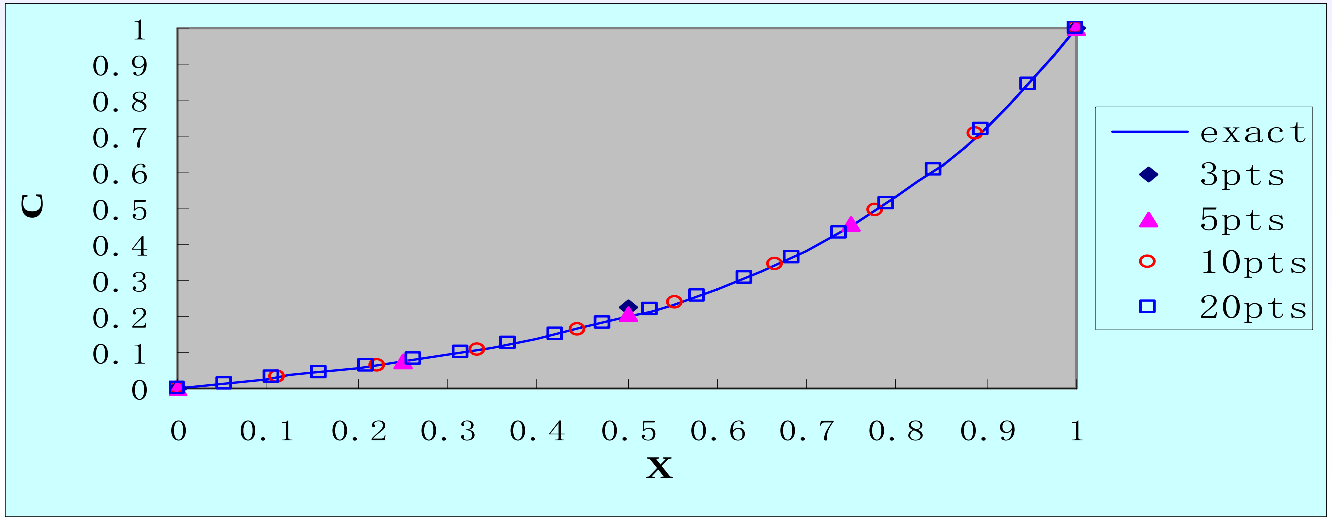
3 POINTS: $X_1=0$ $X_2=0.5$ $X_3=1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -\frac{K\Delta x^2}{D}-2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \Rightarrow C_1 + \left(-\frac{K\Delta x^2}{D} - 2\right) C_2 + C_3 = 0$$

$$C_2 = \frac{D}{K\Delta x^2 + 2D} = \frac{0.01}{0.1*(0.5)^2 + 2*0.01} = 0.222$$

analytical solution

$$C = \frac{1}{e^{\sqrt{K/D}} - e^{-\sqrt{K/D}}} \left(e^{\sqrt{K/D}x} - e^{-\sqrt{K/D}x} \right)$$



FINITE DIFFERENCE METHOD

Example 2: (i)

- Derive a FD approximation for the steady-state reaction-diffusion equation subject to a flux-type boundary condition

$$D\frac{d^2C}{dx^2} - KC = 0, \quad 0 < x < 1cm$$

$$C(0) = 0, \quad D\left.\frac{dC}{dx}\right|_{x=1} = C_*$$

C: concentration, (C_*=0.01 g/cm^3)

D: diffusion coefficient (=0.01 cm^2/s)

K: reaction rate (=0.1 l/s)

Finite diffference approximation for x_{i-1}, x_i and x_{i+1} =>

$$C_{i+1} + \left(-\frac{K\Delta x^2}{D} - 2\right)C_i + C_{i-1} = 0$$

BC: $C_{x=0} = 0$ and

$$\begin{cases} D\dfrac{C_{n+1} - C_{n-1}}{2\Delta x} = C_* & \textit{central difference; or} \\ D\dfrac{C_{n+1} - C_n}{\Delta x} = C_* & \textit{backward difference} \end{cases} \Rightarrow \begin{cases} -C_{n-1} + C_{n+1} = \dfrac{2\Delta x C_*}{D}; \textit{ or} \\ -C_n + C_{n+1} = \dfrac{\Delta x C_*}{D} \end{cases}$$

combined with general FD eq.

$$C_{n-1} + \left(-\frac{K\Delta x^2}{D} - 2\right)C_n + C_{n+1} = 0 \Rightarrow 2C_{n-1} + \left(-\frac{K\Delta x^2}{D} - 2\right)C_n = -\frac{2\Delta x C_*}{D}$$

$$C_{n+1} = \frac{2\Delta x C_*}{D} + C_{n-1}$$

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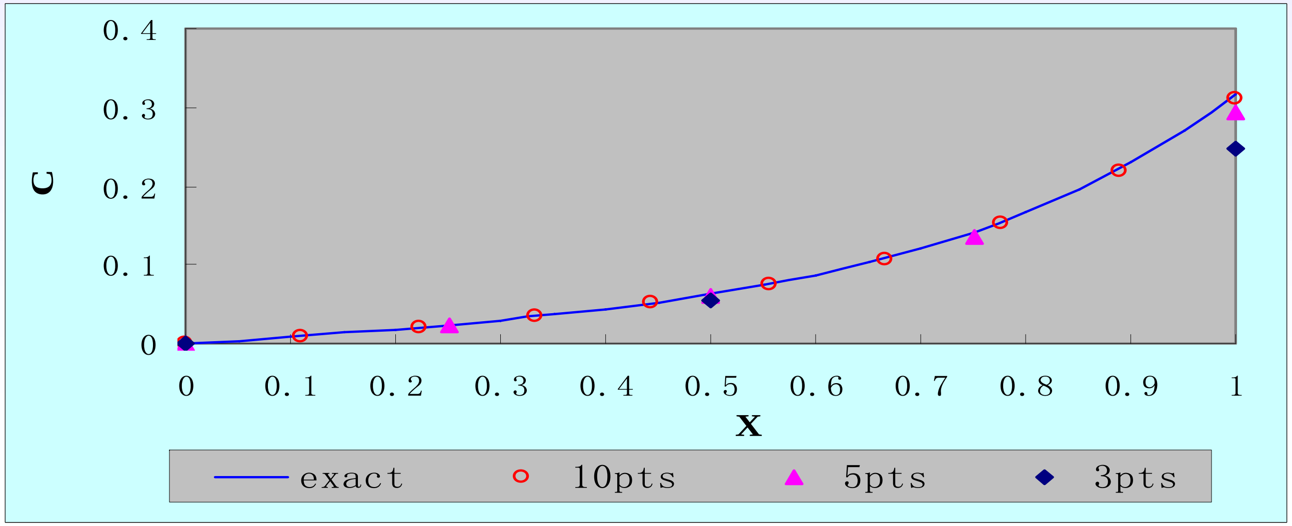
FINITE DIFFERENCE METHOD

Example 2: (ii)

$$
\begin{bmatrix}
1 & 0 & 0 & \dots & \dots & \dots \\
1 & -\frac{K\Delta x^2}{D}-2 & 1 & \dots & \dots & \dots \\
0 & 1 & -\frac{K\Delta x^2}{D}-2 & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & 2 & -\frac{K\Delta x^2}{D}-2
\end{bmatrix}
\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ . \\ C_n \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \\ -\frac{2\Delta x C_*}{D} \end{Bmatrix}
$$

analytical solution

$$
C = \frac{C_*}{D\sqrt{K/D}\left(e^{\sqrt{K/D}} - e^{-\sqrt{K/D}}\right)}\left(e^{\sqrt{K/D}\,x} - e^{-\sqrt{K/D}\,x}\right)
$$



FINITE DIFFERENCE METHOD

Example 3:

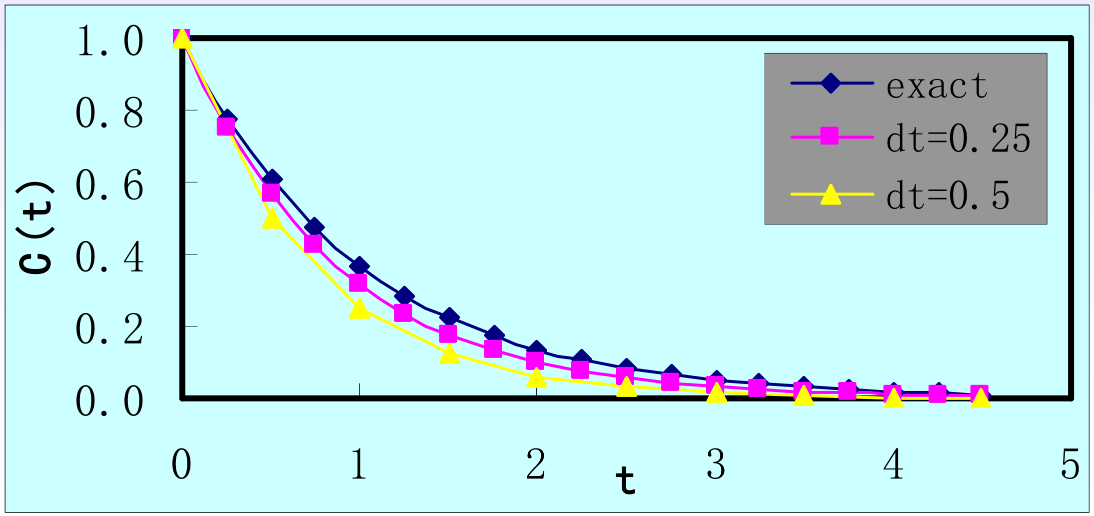
- Use a forward difference method to solve the equation

$$\frac{dC}{dt}+C=0,\quad t>0 \;\; and \;\; C(0)=1$$

Forward difference method for t^k

$$\frac{C^{k+1}-C^k}{\Delta t}+C^k=0 \;\; => \;\; C^{k+1}=(1-\Delta t)C^k$$

Where C^0 is known from the initial condition C(0)=1,



t	exact	dt=0. 25	dt=0.5
0	1. 000	1	1
0. 25	0. 779	0. 75	
0. 5	0. 607	0. 563	0. 5
0. 75	0. 472	0. 422	
1	0. 368	0. 316	0. 25
1. 25	0. 287	0. 237	
1. 5	0. 223	0. 178	0. 125
1. 75	0. 174	0. 133	
2	0. 135	0. 1	0. 0625
2. 25	0. 105	0. 075	
2. 5	0. 082	0. 056	0. 0313
2. 75	0. 064	0. 042	
3	0. 050	0. 032	0. 0156
3. 25	0. 039	0. 024	
3. 5	0. 030	0. 018	0. 0078
3. 75	0. 024	0. 013	
4	0. 018	0. 01	0. 0039
4. 25	0. 014	0. 008	
4. 5	0. 011	0. 006	0. 002

exact solution: $C=e^{-t}$

FINITE DIFFERENCE METHOD

4. Temporal Finite Difference Approximation(i)

- To solve initial value problems,finite difference method is always called on to handle the time derivative.

for the initial value problem $\frac{du}{dt} = f(u, t)$ IC: $u(t_0)=u_0$

result in a system of equations of the for $\left\{\frac{du}{dt}\right\} + [M]\{u\} = \{F\}$

the time derivative is replaced by finite difference approximation

$$\frac{u^{k+1} - u^{k}}{\Delta t} + [M]\{u\} = \{F\}$$

select different time step for (u)

one-step, one-stage

- Explicit method: $\frac{(u^{k+1}) - (u^{k})}{\Delta t} + [M]\{u^{k}\} = \{F\}$
- implicit method: $\frac{(u^{k+1}) - (u^{k})}{\Delta t} + [M]\{u^{k+1}\} = \{F\}$
- weighted method: $\frac{(u^{k+1}) - (u^{k})}{\Delta t} + \theta[M]\{u^{k+1}\} + (1-\theta)[M]\{u^{k}\} = \{F\}$

FINITE DIFFERENCE METHOD

4. Temporal Finite Difference Approximation(ii)

- When θ is selected as 1, the backward method results, while θ =0 is the forward method.

$$\frac{\left(u^{k+1}\right)-\left(u^{k}\right)}{\Delta t}+\theta\left[M\right]\left\{u^{k+1}\right\}+\left(1-\theta\right)\left[M\right]\left\{u^{k}\right\}=\left\{F\right\}$$

- Crank-Nicolson method: θ =0.5
- The first order approximation may have stability problems.

predictor-corrector method:

two-stage, one-step method:

$$u^{*}=u^{k}+\Delta t\left(\left\{F\right\}-\left[M\right]\left\{u\right\}\right)^{k}$$

$$u^{k+1}=u^{k}+\frac{\Delta t}{2}\left(\left(\left\{F\right\}-\left[M\right]\left\{u\right\}\right)^{k}+\left(\left\{F\right\}-\left[M\right]\left\{u^{*}\right\}\right)^{k+1}\right)$$

FINITE DIFFERENCE METHOD

Example 4: (i)

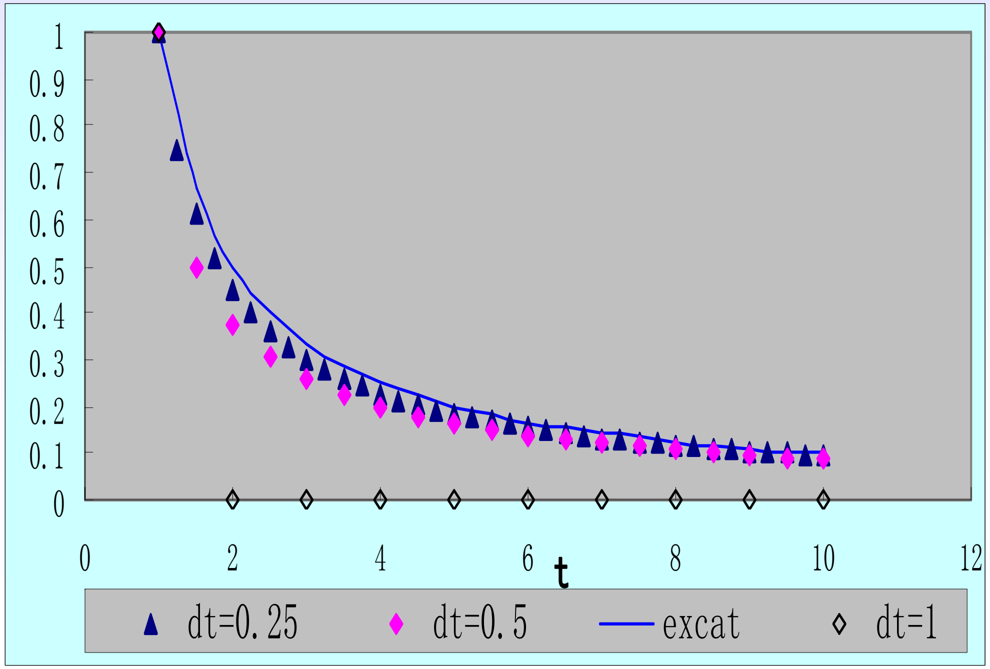
- Solve the non-linear initial value problem

$$\frac{du}{dt} + u^2 = 0, \quad t > 0 \;\; and \;\; u(1) = 1$$

exact solution: $u = \dfrac{1}{t}$

Finite difference approximation

EXPLICIT METHOD: $\dfrac{(u^{k+1}) - (u^k)}{\Delta t} = -(u^k)^2$ => $u^{k+1} = u^k - \Delta t (u^k)^2$



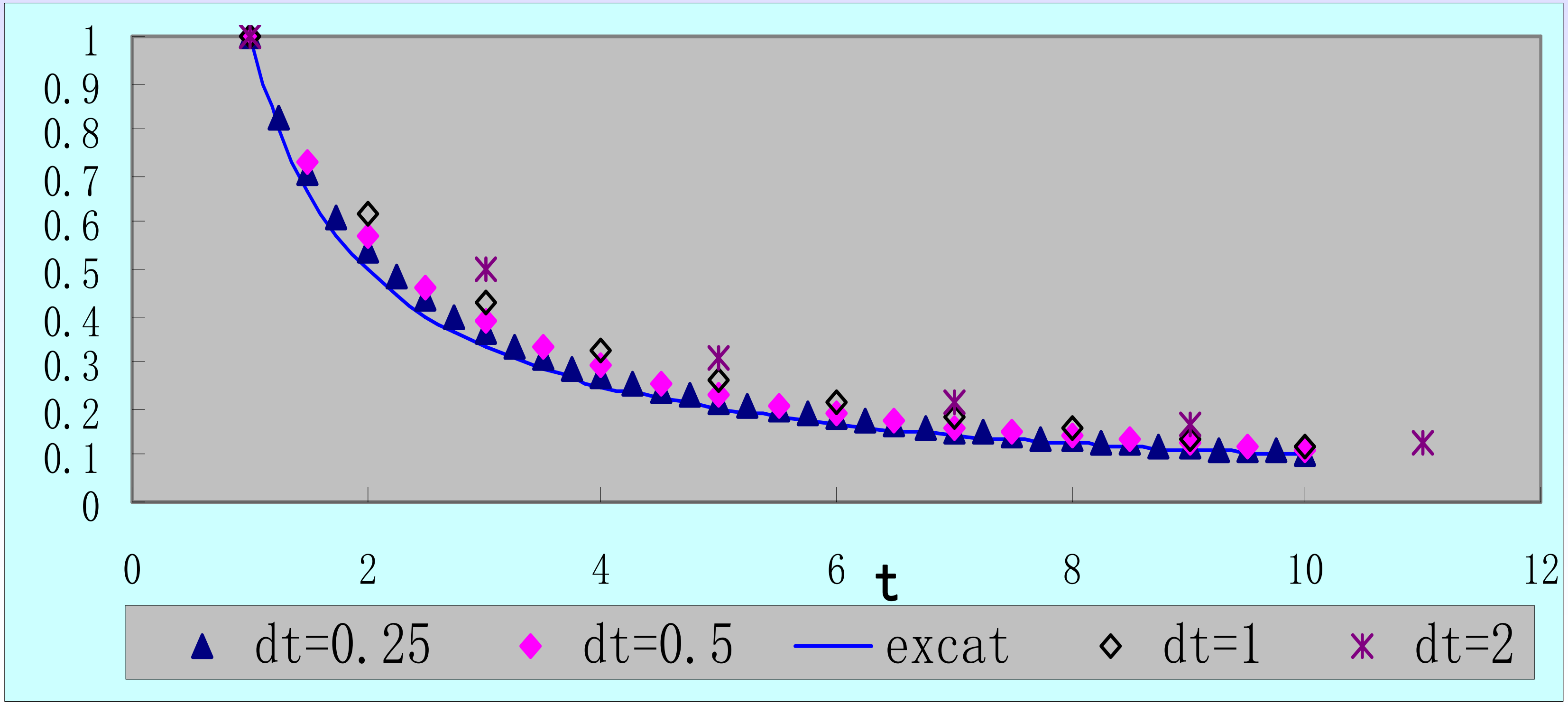
dt=0.5	u	exact	dt=1	u
1	1	1	1	1
1.5	0.5	0.6667	2	0
2	0.375	0.5	3	0
2.5	0.305	0.4	4	0
3	0.258	0.3333	5	0
3.5	0.225	0.2857	6	0
4	0.200	0.25	7	0
4.5	0.180	0.2222	8	0
5	0.164	0.2	9	0
5.5	0.150	0.1818	10	0
6	0.139	0.1667		
6.5	0.129	0.1538		
7	0.121	0.1429		
7.5	0.114	0.1333		
8	0.107	0.125		
8.5	0.101	0.1176		
9	0.096	0.1111		
9.5	0.092	0.1053		
10	0.087	0.1		

FINITE DIFFERENCE METHOD

Example 4: (ii)

- IMPLICIT METHOD:

$$\frac{u^{k+1} - u^{k}}{\Delta t} = -\left(u^{k+1}\right)^2 \quad => \quad u^{k+1} = \frac{-1 + \sqrt{1 + 4\Delta t u^{k}}}{2\Delta t}$$



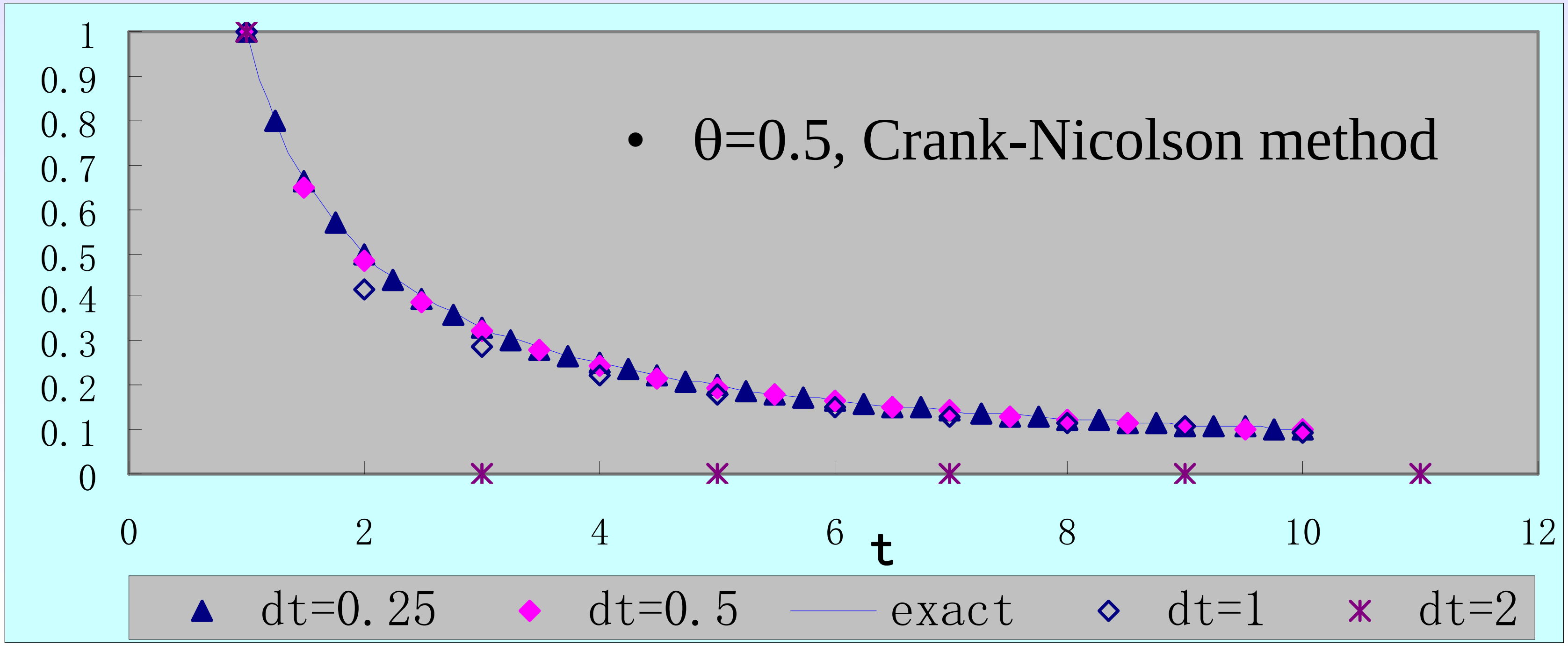
FINITE DIFFERENCE METHOD

Example 4: (iii)

- WEIGHTED METHOD:

Finite difference approximation

$$\frac{\left(u^{k+1}\right)-\left(u^{k}\right)}{\Delta t}=-\theta\left(u^{k+1}\right)^2-(1-\theta)\left(u^{k}\right)^2 \quad => \quad u^{k+1}=\frac{-1+\sqrt{1-4\theta\Delta t\left((1-\theta)\Delta t\left(u^{k}\right)^2-u^{k}\right)}}{2\theta\Delta t}$$

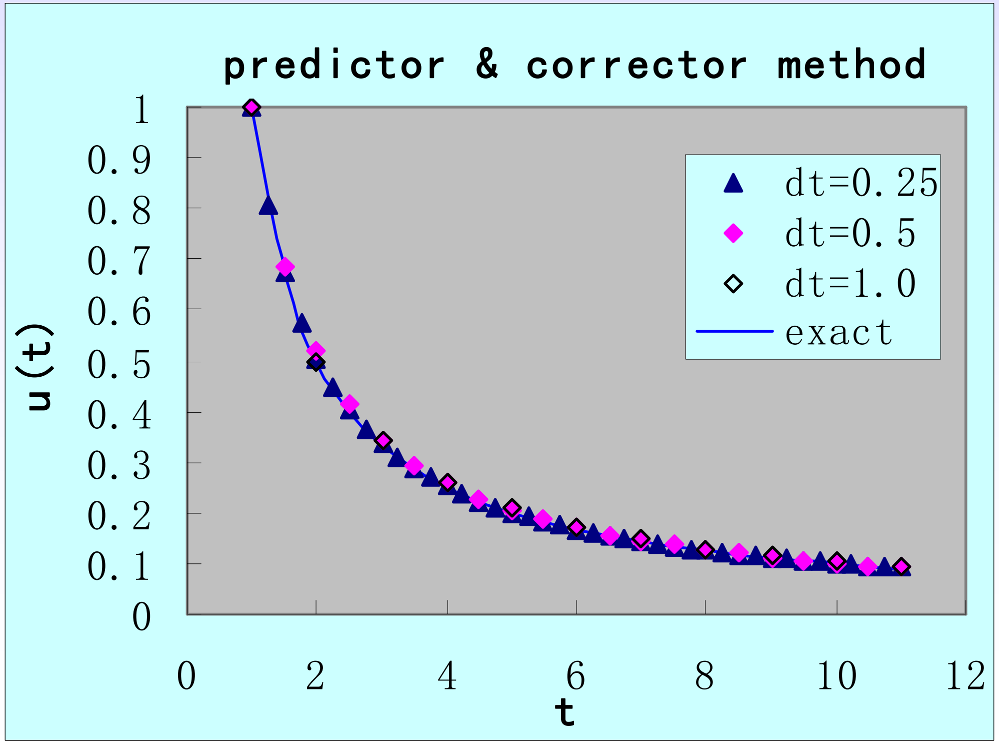


FINITE DIFFERENCE METHOD

Example 4: (iv)

- Predictor-corrector method

$$\left.\begin{aligned} u^* &= u^k - \Delta t \left(u^k\right)^2 \\ u^{k+1} &= u^k - \frac{\Delta t}{2}\left[\left(u^k\right)^2 + \left(u^*\right)^2\right] \end{aligned}\right\}$$

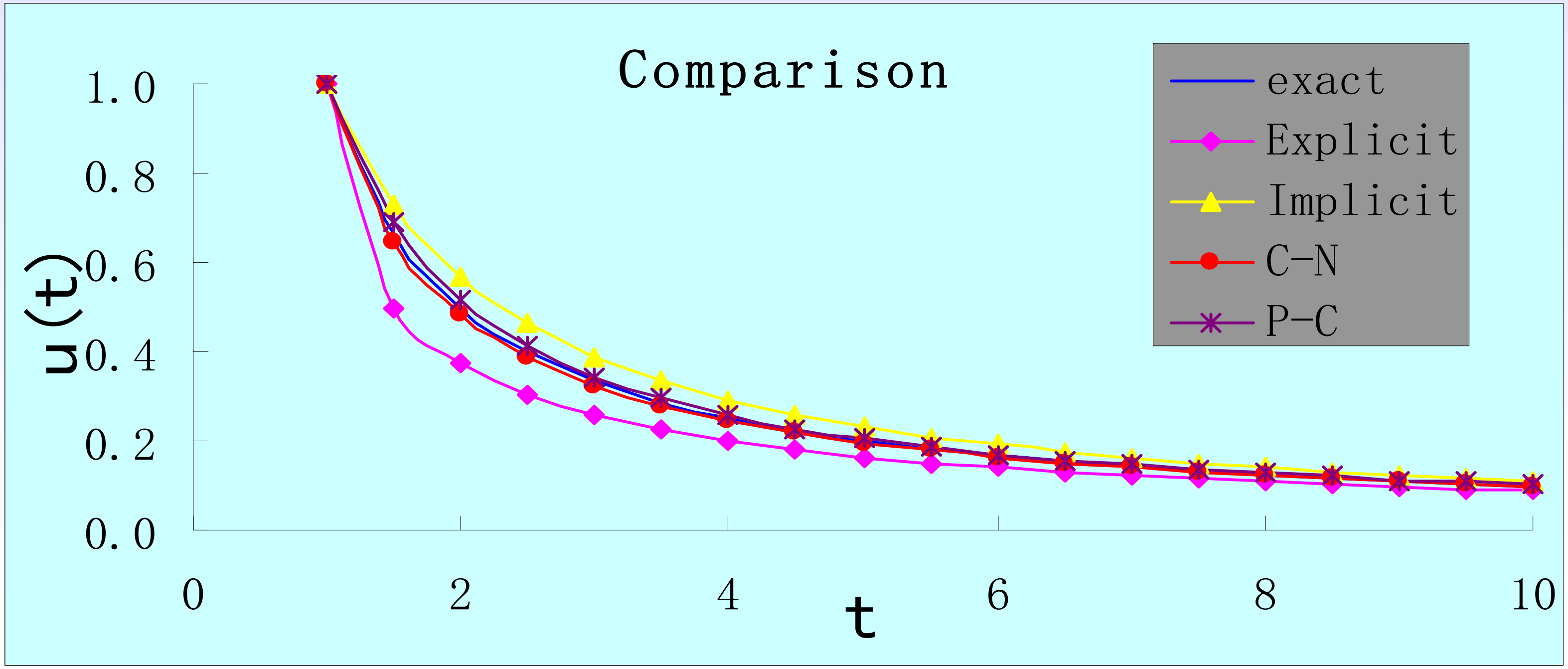


	N	O	P	Q
5	t	u	u*	exact
6	1	1		1
7	1.5	0.6875	0.5	0.6667
8	2	0.5184	0.4512	0.5
9	2.5	0.4144	0.3841	0.4
10	3	0.3445	0.3285	0.3333
11	3.5	0.2945	0.2851	0.2857
12	4	0.257	0.2511	0.25
13	4.5	0.228	0.224	0.2222
14	5	0.2048	0.202	0.2
15	5.5	0.1858	0.1838	0.1818
16	6	0.1701	0.1686	0.1667
17	6.5	0.1568	0.1556	0.1538
18	7	0.1454	0.1445	0.1429
19	7.5	0.1356	0.1349	0.1333
20	8	0.127	0.1264	0.1250
21	8.5	0.1194	0.119	0.1176
22	9	0.1127	0.1123	0.1111
23	9.5	0.1067	0.1064	0.1053
24	10	0.1013	0.101	0.1
25	10.5	0.0964	0.0962	0.0952
26	11	0.092	0.0918	0.0909

FINITE DIFFERENCE METHOD

Example 4: (v)

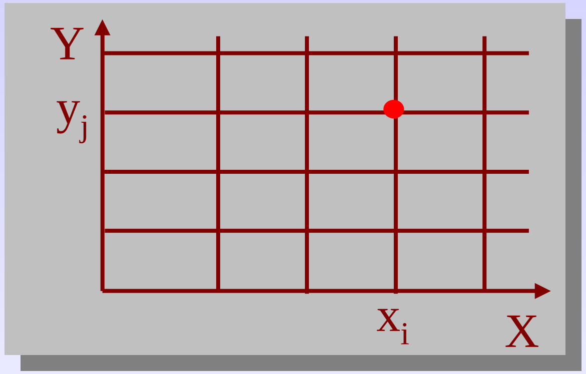
- Compare results from 4 different methods (Δt=0.5)
  - —computational effort
  - —computer storage
  - —accuracy
  - —stability



FINITE DIFFERENCE METHOD

Example 5: (i)

- Obtain a finite difference solution for the 2D problem.



$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=-\pi^2\sin(\pi x)\sin(\pi y),\ \ for\quad 0\le x\le 1;\ 0\le y\le 1$$

$$h(0,y)=1,\ h(1,y)=y\qquad h(x,0)=1-x,\ h(x,1)=1$$

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=-\frac{R(x,y)}{T}$$ ← GW flow with recharge R

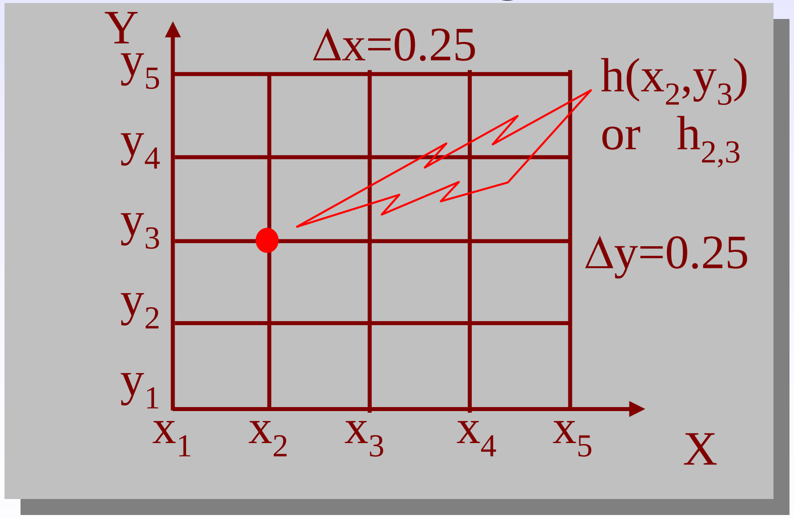
Step 1: select the grid system

$$let\quad \Delta x=\Delta y=0.25$$

Step 2: finite difference method

$$\frac{\partial^2 h}{\partial x^2}=\frac{h_{i-1,j}-2h_{i,j}+h_{i+1,j}}{\Delta x^2}+O(\Delta x^2)$$

$$\frac{\partial^2 h}{\partial y^2}=\frac{h_{i,j-1}-2h_{i,j}+h_{i,j+1}}{\Delta y^2}+O(\Delta y^2)$$



$$\frac{h_{i-1,j}-2h_{i,j}+h_{i+1,j}}{\Delta x^2}+\frac{h_{i,j-1}-2h_{i,j}+h_{i,j+1}}{\Delta y^2}=f(x_i,y_j)$$

$$f(x_i,y_j)=-\pi^2\sin(\pi x_i)\sin(\pi y_j)$$

FINITE DIFFERENCE METHOD

Example 5: (ii)

- Application of these equations at each interior node leads to nine independent linear algebraic equation, each of which has the form.

$$h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1} - 4h_{i,j} = \Delta x^2 f\left(x_i, y_j\right)$$

Step 3: form the matrix for nodes (2,2), (2,3), (2,4), (3,2), (3,3), (3,4), (4,2), (4,3) and (4,4)

$$h_{1,2} + h_{3,2} + h_{2,1} + h_{2,3} - 4h_{2,2} = \Delta x^2 f\left(x_2, y_2\right)$$

$$h_{1,3} + h_{3,3} + h_{2,2} + h_{2,4} - 4h_{2,3} = \Delta x^2 f\left(x_2, y_3\right)$$

$$h_{1,4} + h_{3,4} + h_{2,3} + h_{2,5} - 4h_{2,4} = \Delta x^2 f\left(x_2, y_4\right)$$

$$h_{2,2} + h_{4,2} + h_{3,1} + h_{3,3} - 4h_{3,2} = \Delta x^2 f\left(x_3, y_2\right)$$

$$h_{2,3} + h_{4,3} + h_{3,2} + h_{3,4} - 4h_{3,3} = \Delta x^2 f\left(x_3, y_3\right)$$

$$h_{2,4} + h_{4,4} + h_{3,3} + h_{3,5} - 4h_{3,4} = \Delta x^2 f\left(x_3, y_4\right)$$

$$h_{3,2} + h_{5,2} + h_{4,1} + h_{4,3} - 4h_{4,2} = \Delta x^2 f\left(x_4, y_2\right)$$

$$h_{3,3} + h_{5,3} + h_{4,2} + h_{4,4} - 4h_{4,3} = \Delta x^2 f\left(x_4, y_3\right)$$

$$h_{3,4} + h_{5,4} + h_{4,3} + h_{4,5} - 4h_{4,4} = \Delta x^2 f\left(x_4, y_4\right)$$

$$h_{3,2} + h_{2,3} - 4h_{2,2} = \Delta x^2 f\left(x_2, y_2\right) - h_{2,1} - h_{1,2}$$

$$h_{3,3} + h_{2,2} + h_{2,4} - 4h_{2,3} = \Delta x^2 f\left(x_2, y_3\right) - h_{1,3}$$

$$h_{3,4} + h_{2,3} - 4h_{2,4} = \Delta x^2 f\left(x_2, y_4\right) - h_{1,4} - h_{2,5}$$

$$h_{2,2} + h_{4,2} + h_{3,3} - 4h_{3,2} = \Delta x^2 f\left(x_3, y_2\right) - h_{3,1}$$

$$h_{2,4} + h_{4,4} + h_{3,3} - 4h_{3,4} = \Delta x^2 f\left(x_3, y_4\right) - h_{3,5}$$

$$h_{3,2} + h_{4,3} - 4h_{4,2} = \Delta x^2 f\left(x_4, y_2\right) - h_{5,2} - h_{4,1}$$

$$h_{3,3} + h_{4,2} + h_{4,4} - 4h_{4,3} = \Delta x^2 f\left(x_4, y_3\right) - h_{5,3}$$

$$h_{3,4} + h_{4,3} - 4h_{4,4} = \Delta x^2 f\left(x_4, y_4\right) - h_{5,4} - h_{4,5}$$

FINITE DIFFERENCE METHOD

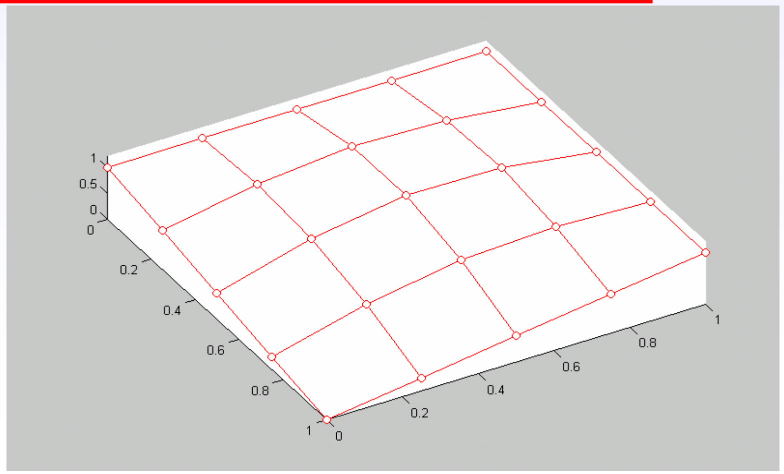
Example 5: (iii)

- The matrix to be solved is

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{Bmatrix} h_{2,2} \\ h_{2,3} \\ h_{2,4} \\ h_{3,2} \\ h_{3,3} \\ h_{3,4} \\ h_{4,2} \\ h_{4,3} \\ h_{4,4} \end{Bmatrix} = \begin{Bmatrix} -2.058 \\ -1.436 \\ -2.308 \\ -0.936 \\ -0.617 \\ -1.436 \\ -0.808 \\ -0.936 \\ -2.058 \end{Bmatrix}$$

result:

$$\begin{Bmatrix} h_{2,2} \\ h_{2,3} \\ h_{2,4} \\ h_{3,2} \\ h_{3,3} \\ h_{3,4} \\ h_{4,2} \\ h_{4,3} \\ h_{4,4} \end{Bmatrix} = \begin{Bmatrix} 1.076 \\ 1.247 \\ 1.201 \\ 0.997 \\ 1.276 \\ 1.247 \\ 0.701 \\ 0.997 \\ 1.076 \end{Bmatrix}$$



FINITE DIFFERENCE METHOD

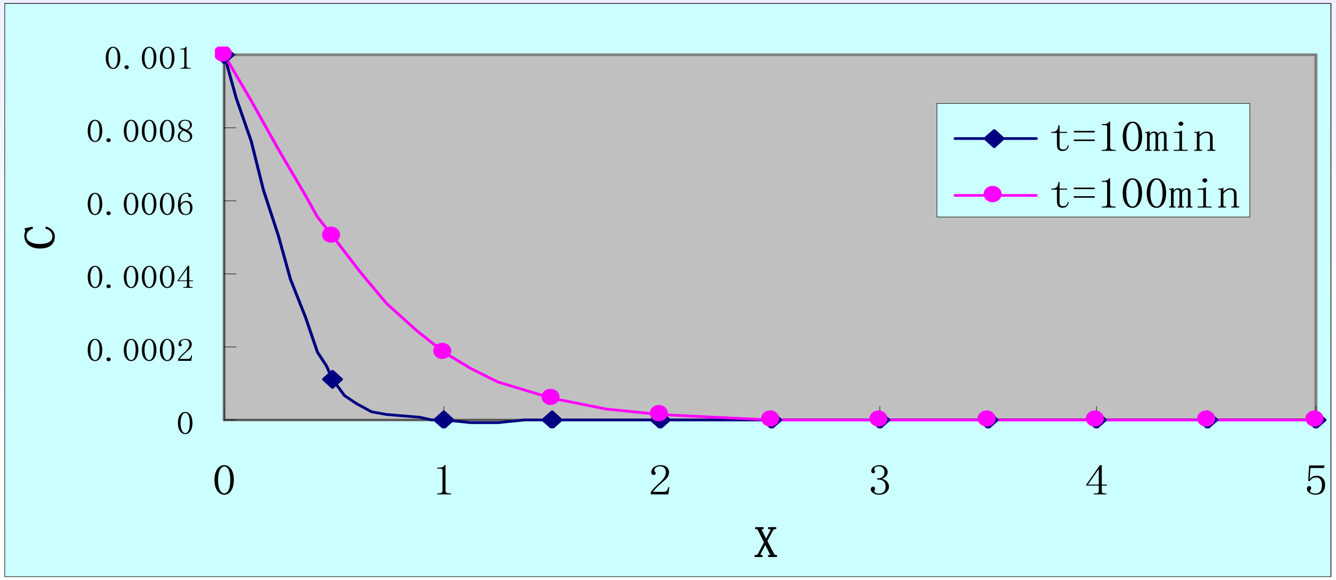
Example 6: (i)

- Solve $\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$ $C(0,t) = C_0,\ C(\infty,t) = 0,\ and\ \ C(x,0) = 0$ use C_0=1 mg/m^3 ,D=5×10^{-5} m^2/s in the calculation

Fully explicit finite difference approximation is

$$\frac{C_i^{k+1} - C_i^k}{\Delta t} = D\frac{C_{i+1}^k - 2C_i^k + C_{i-1}^k}{\Delta x^2} \Rightarrow C_i^{k+1} = \frac{D\Delta t}{\Delta x^2}C_{i+1}^k + \left(1 - \frac{2D\Delta t}{\Delta x^2}\right)C_i^k + \frac{D\Delta t}{\Delta x^2}C_{i-1}^k$$

From the condition $\gamma = \frac{D\Delta t}{\Delta x^2} \le \frac{1}{2}$, select Δx=0.5m, Δt=5 min



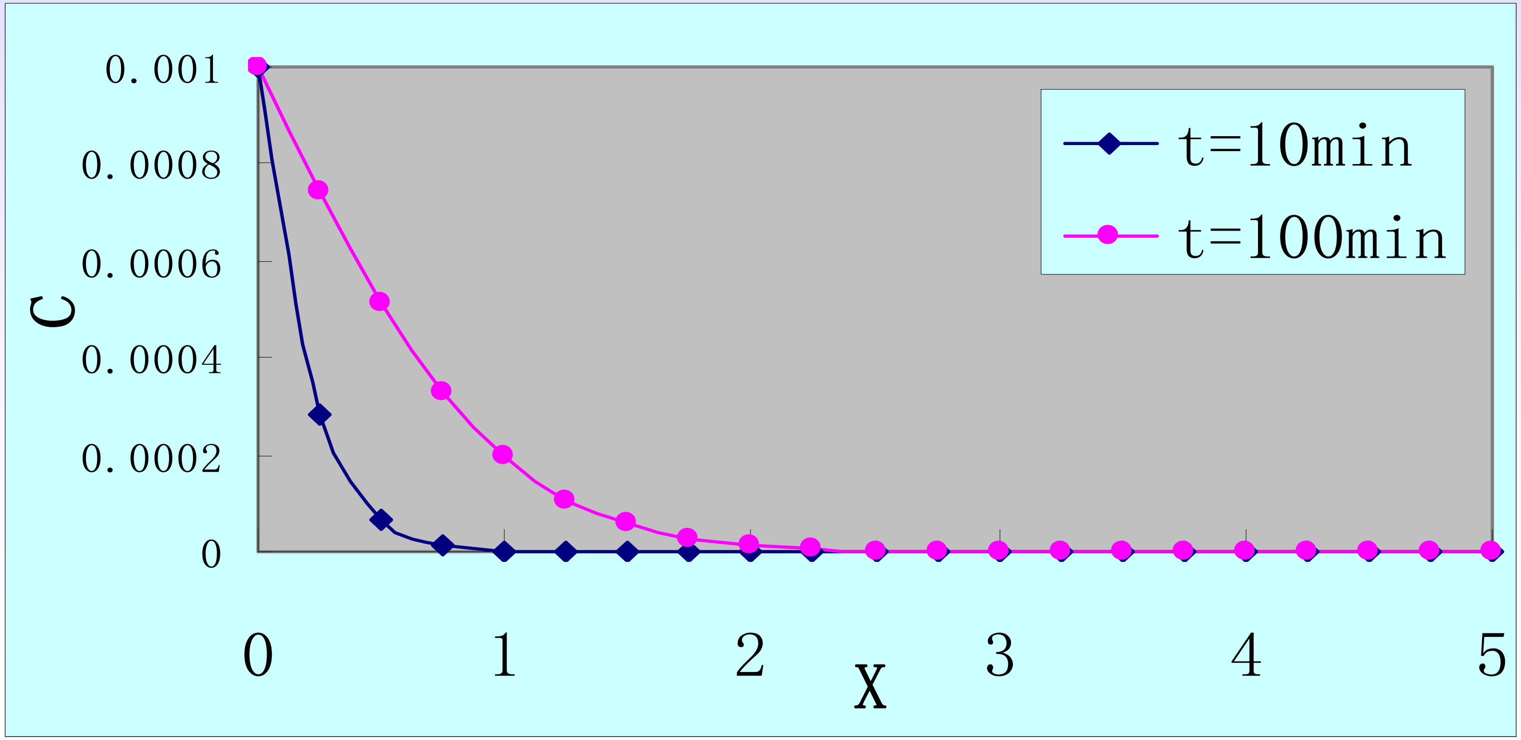
FINITE DIFFERENCE METHOD

Example 6: (ii)

- Solve the problem with fully implicit method

$$\frac{C_i^{k+1}-C_i^k}{\Delta t}=D\frac{C_{i+1}^{k+1}-2C_i^{k+1}+C_{i-1}^{k+1}}{\Delta x^2} \Rightarrow \frac{D\Delta t}{\Delta x^2}C_{i+1}^{k+1}+\left(-1-\frac{2D\Delta t}{\Delta x^2}\right)C_i^{k+1}+\frac{D\Delta t}{\Delta x^2}C_{i-1}^{k+1}=-C_i^k$$

select Δx=0.25m, Δt=10 min and 100min



Thanks